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Step 1

Part a: Lewis algorithm based European option pricing with calibrated Heston model for 15D to expire

We observe the 15D-to-expire European options market as below

DaysToMaturity	Strike	CALL	PUT	T	r
15	227.5	10.52	4.32	0.06	0.015
15	230	10.05	5.2	0.06	0.015
15	232.5	7.75	6.45	0.06	0.015
15	235	6.01	7.56	0.06	0.015
15	237.5	4.75	8.78	0.06	0.015

Note, we convert "DaysToMaturity" to annualised expiry "T" via dividing with 250 business-days as the dates in the data original are labelled with days not date. Typical tenor annualisation should refer to the market convention, for instance the European rates market follows 30/360 rule where one should divide the number of days by 360 instead of 365 and 365 for americans. That counting also requires the holiday calendar to extract actual business days. Therefore, the 250 BDs per annum convention is the only applicable unit applicable here.

To price with the fitted Heston model, we calibrate the Heston model with below configuration:

- Pricing method: Lewis method that numerically integrates the characterisfic function of the Heston process with CALL-PUT price parity to calculate PUT price from the CALL price via $P = C - S0 + K \cdot e^{-r \cdot T}$
- Target option type: European
- Target option side: Each CALL and PUT
- Price data points: Those at 15D days-to-maturity on the same side
- Calibration objective: Minimise the `Mean-Squared-Error` (MSE - squared price unit) between the market and model option prices
- Calibration procedure:
  1. Brute-forcing the heston parameters over a grid (via scipy.opt.brute),
  2. Nelder-Mead (via scipy.opt.fmin) from the optimal brute-force result upto error convergence or max iteration

The separation of the CALL and PUT as separate model targets to calibrate is to adopt the option market smirk as shown in the data.

As a result of calibration, we got

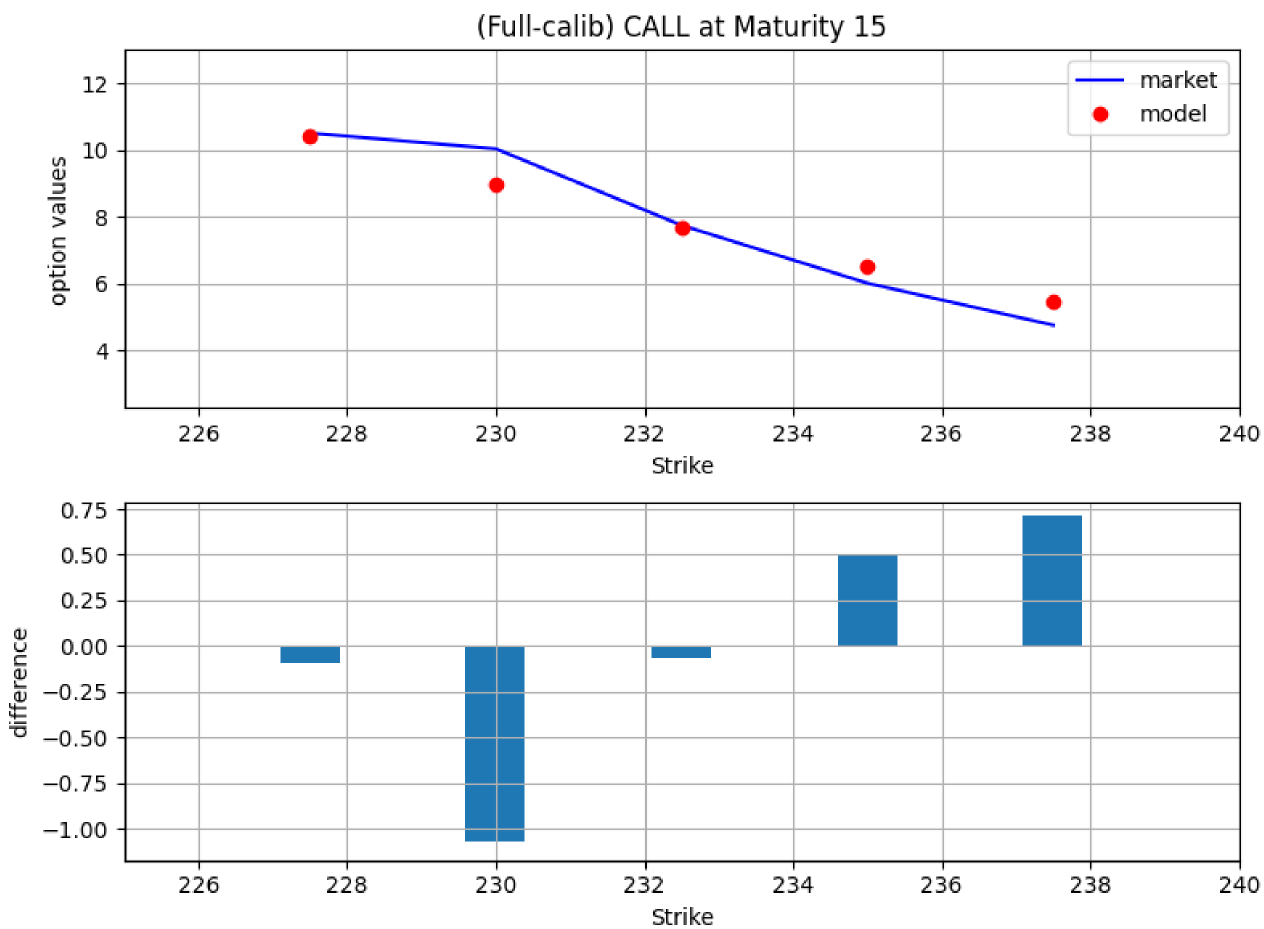
S0	r	kappa	theta	sigma	rho	v0
232.9	0.015	4.00618e-08	0.124817	0.000100004	0.614373	0.103351

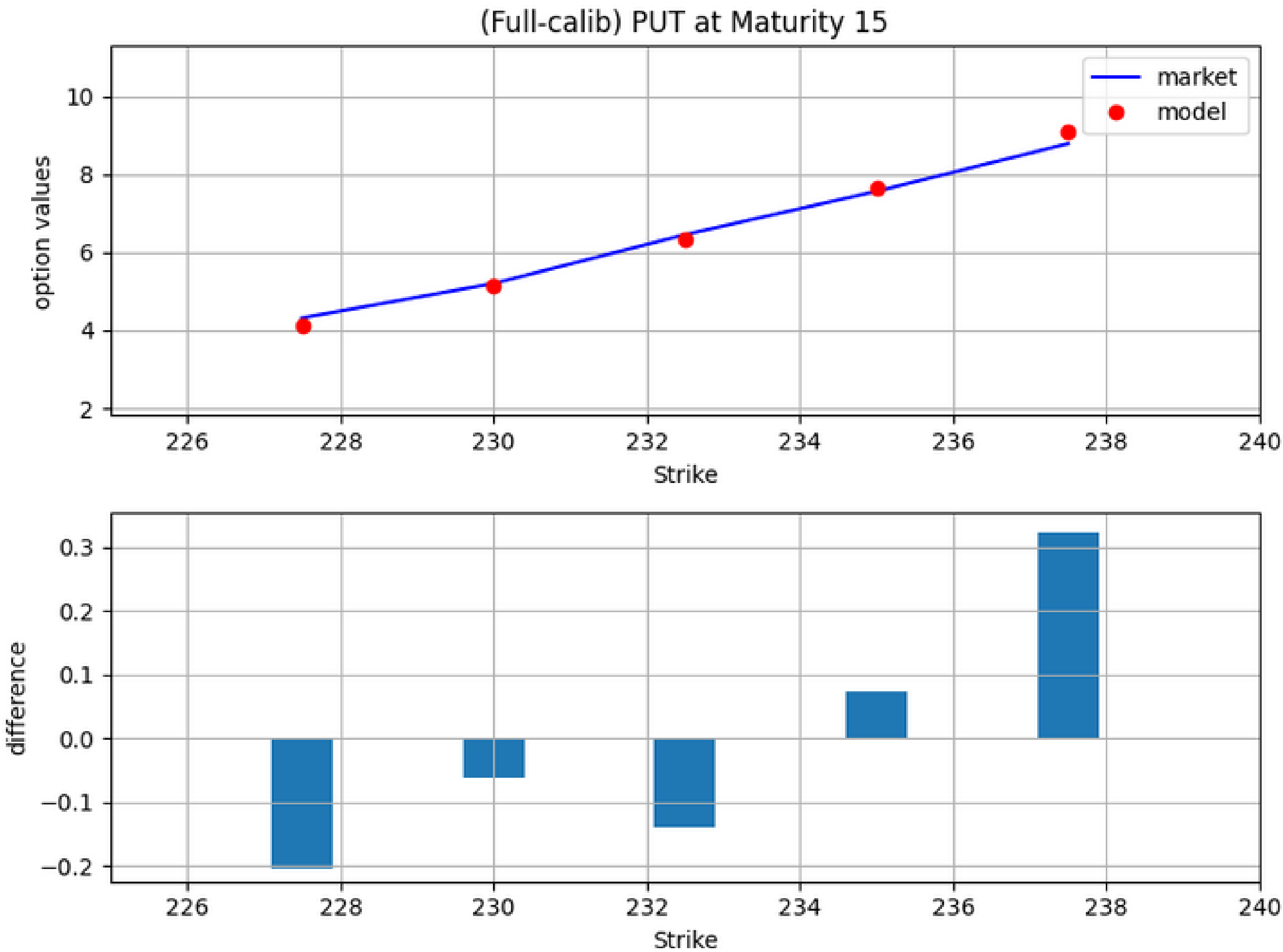
for the CALL options with MSE at 0.378 price mark squared, and

S0	r	kappa	theta	sigma	rho	v0
232.9	0.015	3.96864e-07	0.097427	0.000278084	-0.174123	0.0846418

for the PUT options with MSE at 0.035 mark.

Below are the model estimated option price versus the market price.





Part b: Carr-Madan algorithm based European option pricing with calibrated Heston model for 15D to expire

The practice here does not differ any from the part a's, except the option price calculation logic adopt the Carr-Madan's method to integrate over the characteristic function of the Heston dynamics instead of Lewis's. If the price outputs from this method aligns with the outputs in the part a's, then we can indirectly cross-check the EUR option pricing validity.

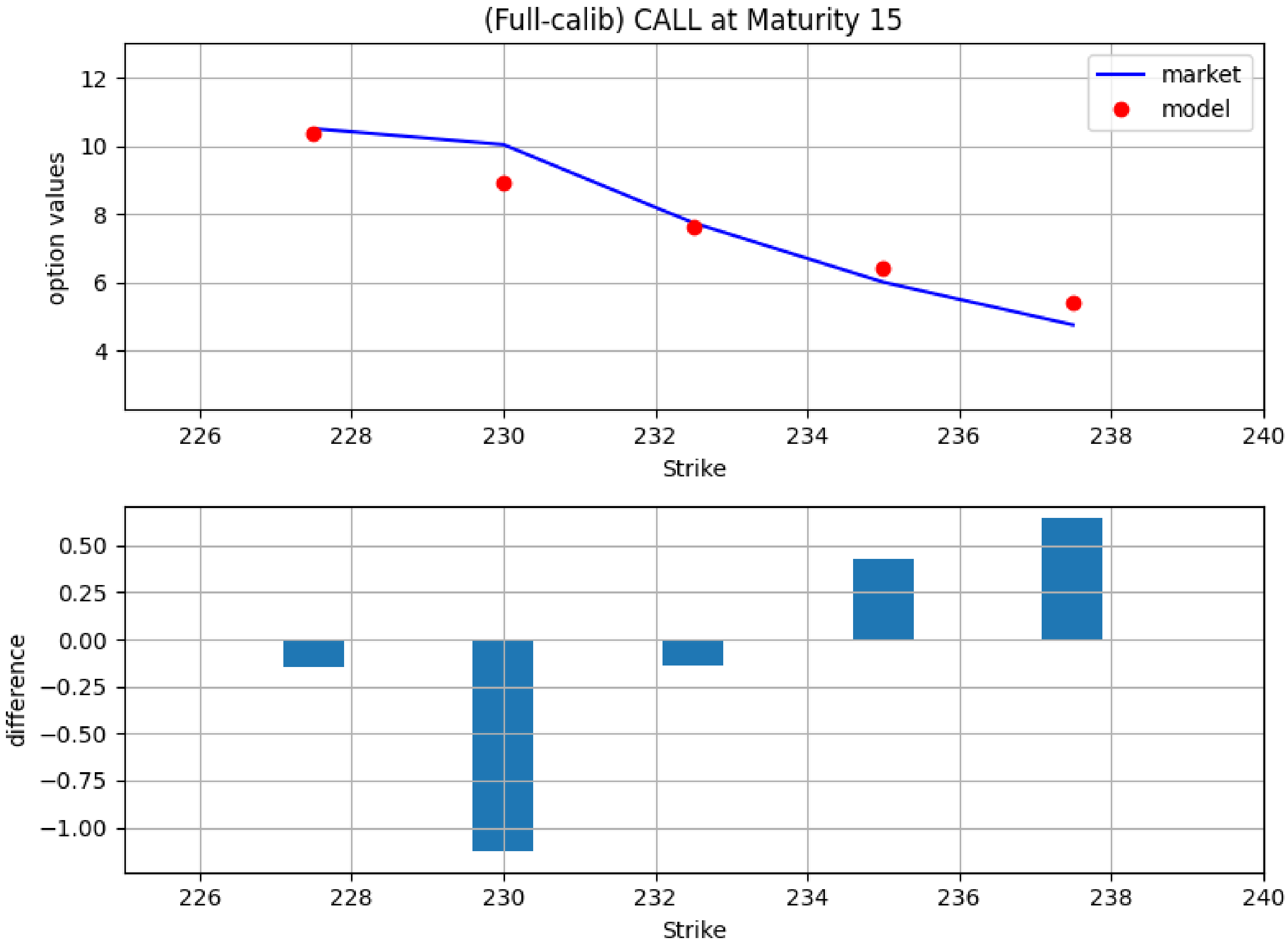
CALL side calibration result with MRE 0.378:

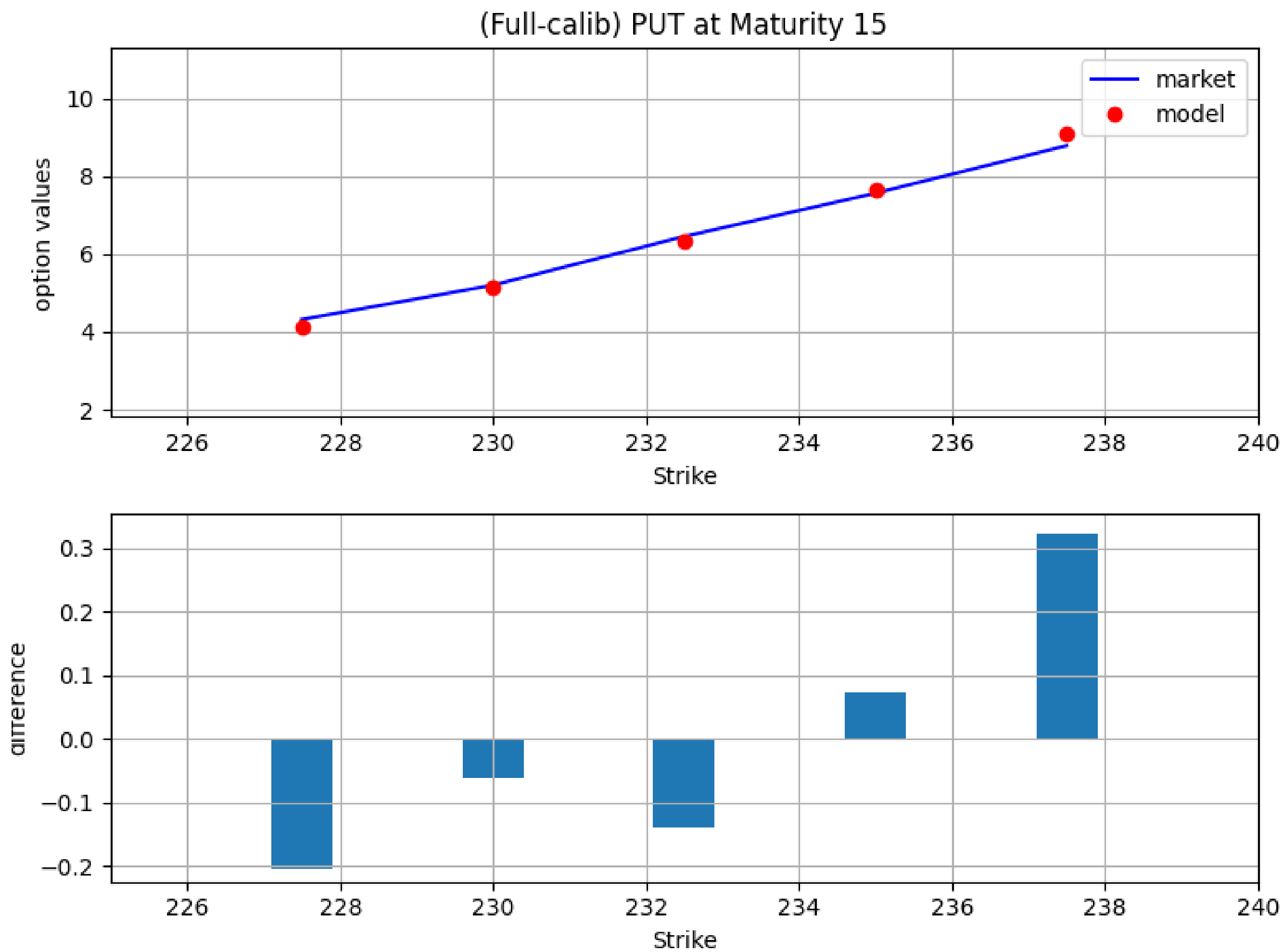
S0	r	kappa	theta	sigma	rho	v0
232.9	0.015	4.00806e-08	0.124816	0.000100027	0.614291	0.10335

PUT side calibration result with MRE 0.035:

S0	r	kappa	theta	sigma	rho	v0
232.9	0.015	5.85713e-08	0.08543	0.000100037	0.240845	0.084424

We can observe the MREs as well as the calibrated parameters assimilate to the ones we saw in the part except ρ on the PUT side where the sign has flipped.





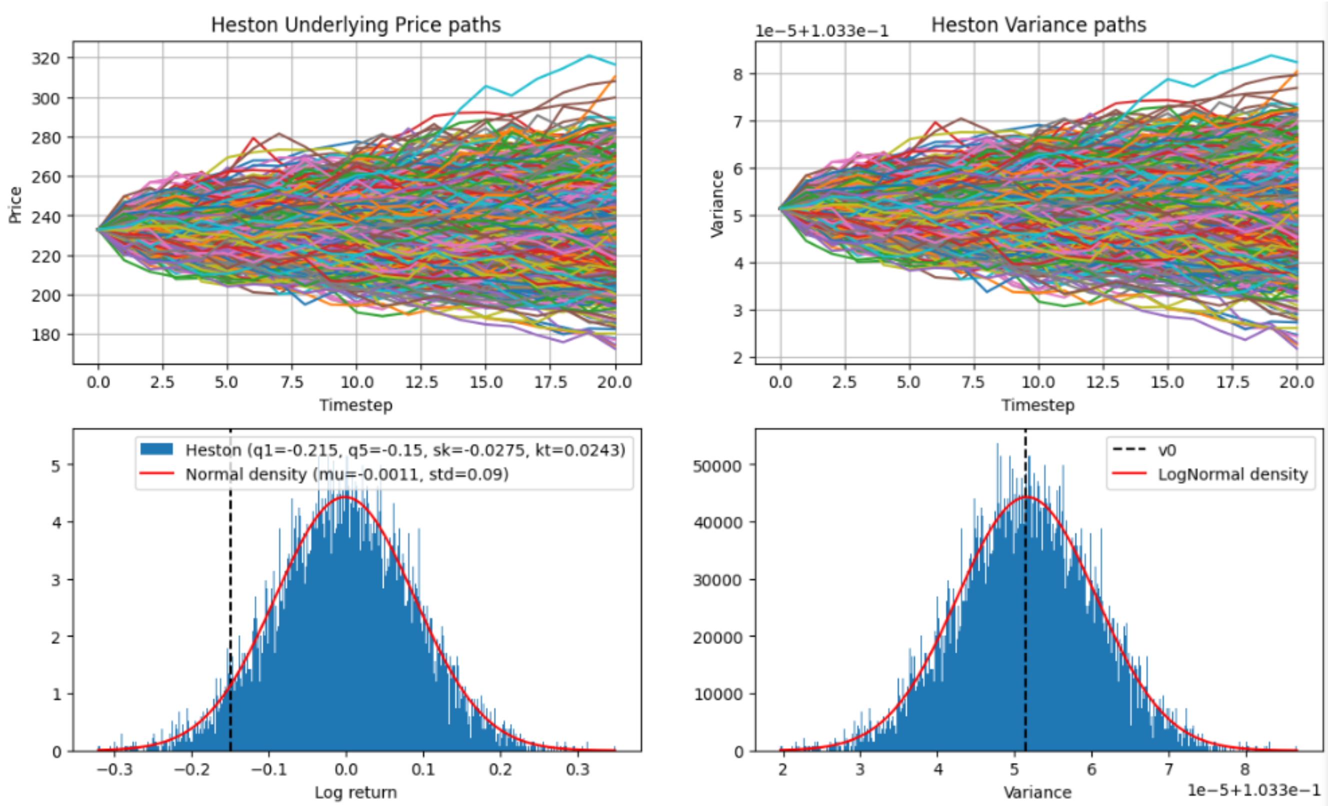
part c: Asian CALL options pricing via calibrated Heston model with Lewis pricing algorithm

We sampled paths with below formula - underlying as the integral form to apply the volatility adjustment effect on drift for log-return process, and naive Euler-Maruyama scheme for the variance process:

$$X_{t+\Delta t} = X_t \cdot \exp\left[(r - 0.5\nu_t)t + \sqrt{\nu_t}\Delta W_t^{(1)}\right]$$
$$\nu_{t+\Delta t} = \nu_t + \kappa \cdot (\theta - \nu_t)dt + \sigma\sqrt{\nu_t}\Delta W_t^{(2)}$$
$$\text{where } d < W^{(1)}, W^{(2)} >_t = \rho \cdot dt$$

Euler-Maruyama scheme is a naive implementation of the SDE form of the process, which is still applicable for the variance process as it is not the logreturn process which requires the drift adjustment.

From the calibrated Heston model (on CALL side) achieved in the part a, we run a monte-carlo simulation of 20 steps (= number of days to expire) over 10000 paths as shown below.



The resultant Asian CALL option price around the ATM (moneyness 0.95, 0.98, 1.0, 1.02, 1.05) is a shown below with 4% charged offer price on top of the fair price:

Moneyness	Fair price	Our Offer
0.95	12.844958001020133	13.3587563211
0.98	7.590076359084218	7.89367941345
1.0	4.91738550547796	5.1140809257
1.02	2.963940058821264	3.08249766117
1.05	1.209679455108908	1.25806663331

Step2

part a: European option pricing with calibrated Bates model for 60D to expire with Lewis method

We observe the 60D-to-expire European options market as below

DaysToMaturity	Strike	CALL	PUT	T	r
60	227.5	16.78	11.03	0.164384	0.015
60	230	17.65	12.15	0.164384	0.015
60	232.5	16.86	13.37	0.164384	0.015
60	235	16.05	14.75	0.164384	0.015
60	237.5	15.1	15.62	0.164384	0.015

Below is the calibration procedure used in this pricing practice:

- Pricing method: Lewis method with CALL-PUT price parity
- Target option type: European
- Target option side: Each CALL and PUT
- Price data points: Those at 60D days-to-maturity on the same side
- Calibration objective: Minimise the `Mean-Squared-Error` (MSE - squared price unit) between the market and model option prices. Parameters space grid brute-forcing and then simplex downhill from the lowest point from the grid.
- Calibration procedure:
  1. Calibrate the Heston parameters using the Lewis pricing algorithm for the Heston dynamics model.
  2. Calibrate the jump parameters using the Lewis pricing algorithm for the Bates dynamics. Variables to calibrate are the ones in the Merton model except the noise diffusion parameter as the drift is governed under the Heston dynamics. Heston parameters are given from the previous step.
  3. Calibrate the whole parameters (Heston + Merton jump only) with the simplex downhill from the parameters point fitted in the last two steps

As a result of calibration, we got

S0	r	lambd	mu	delta	kappa	theta	sigma	rho	v0
232.9	0.015	0.514295	-8.83544e-13	0.99054	1.9766e-08	0.277956	0.000104824	-0.0184529	0.0153145

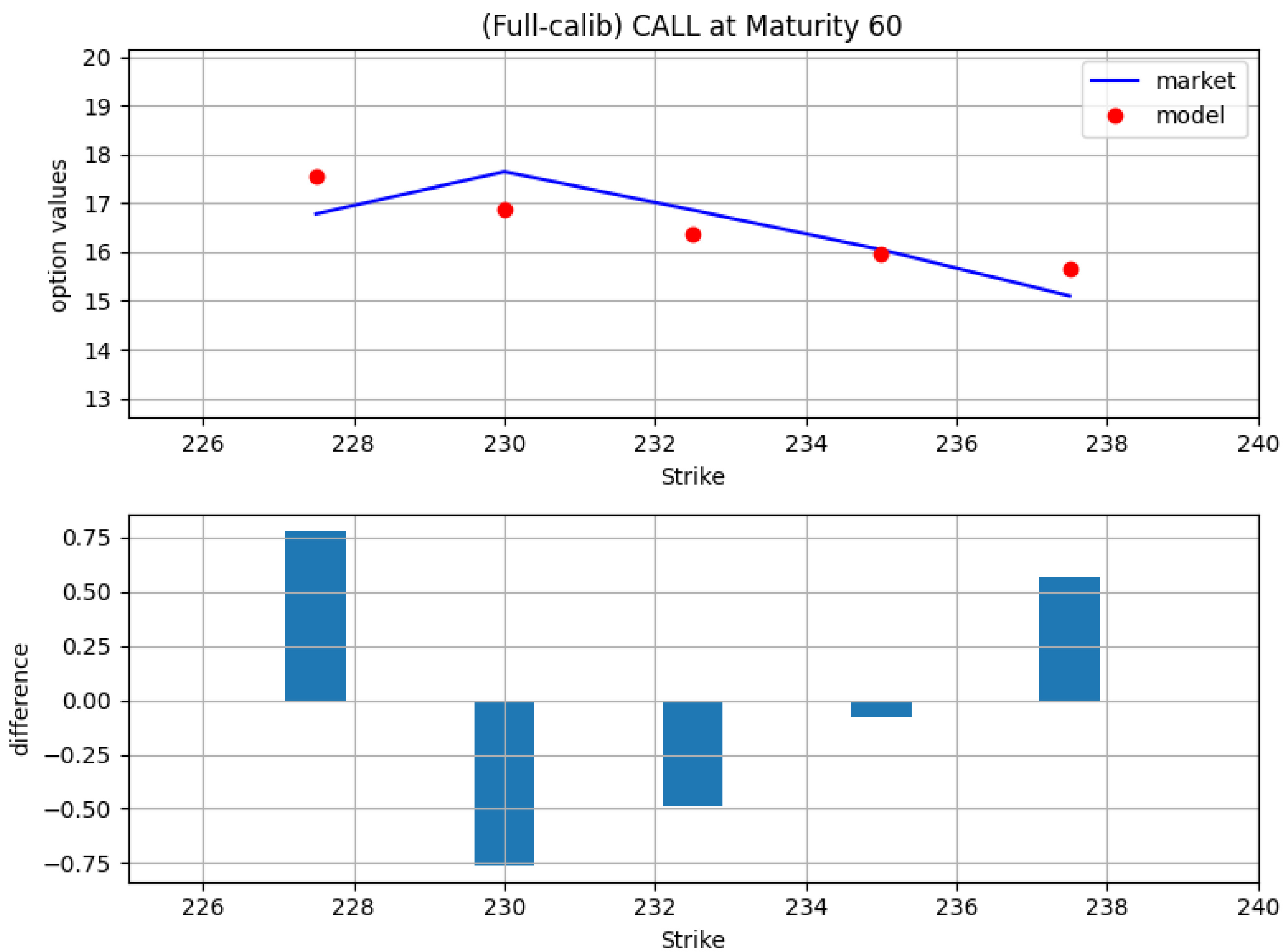
for the CALL options with MSE at 0.349 price mark squared, and

S0	r	lambd	mu	delta	kappa	theta	sigma	rho	v0
232.9	0.015	1.65191	-0.112154	0.396551	2.84644e-08	0.176102	0.000100126	0.70556	0.0242659

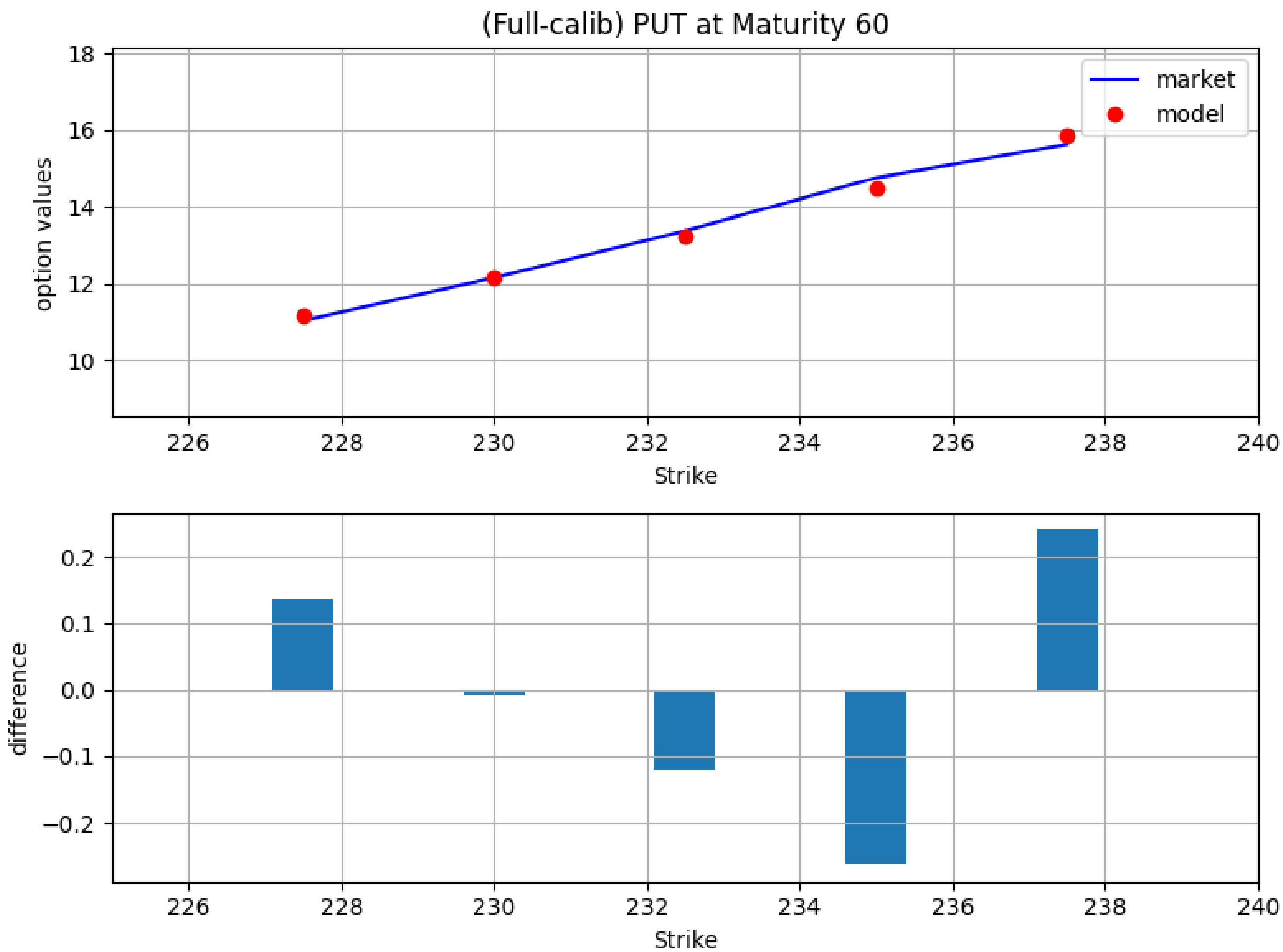
for the PUT options with MSE at 0.032 mark.

This is a few percent improvement on the MSE upto 10%, not game-changingly significant but still not ignorably marginal. The big difference in the calibrated parameters are from λ, μ, δ where the PUT's model assume more frequent, weaker jumps with higher shift moderation and by ρ being significanly positive the dynamics shall exploit longer self-excitation regime as positive variance couples with positive noise multiplier. With the extremely low κ, the window of self-excitation shall rule for long period of time as decay doesn't almost occur.

This presents the semantics of the option smirk that the PUT prices are relatively **lower** than that of CALL at the same distance from the ATM: PUT dynamics assume higher underlying price hence price less. Below are the model estimated option price versus the market price. The calibration quality is not satisfying to be fair, but we have kink in the market price in the data where the stochastic process (at least those we have discussed) do not have good properties to calibrate agains, and taming the multi-dimensional fitting is a realm of art with *good* search grid apriori is required where we did not have a good luck in limited time of this study.



(Full-calib) CALL at Maturity 60
market
model
option values
Strike
difference
Strike



part b: European option pricing with calibrated Bates model for 60D to expire with Carr-Madan method

The procedure is the same as that of part a, but the option pricing algorithm is Carr-Madan's method instead of Lewis's method offering the faster calculation speed of the option price via the Fourier-Transform inversion of the option price's characteristic function.

As a result of calibration, we got

S0	r	lambd	mu	delta	kappa	theta	sigma	rho	v0
232.9	0.015	0.514295	-8.83544e-13	0.99054	1.9766e-08	0.277956	0.000104824	-0.0184529	0.0153145

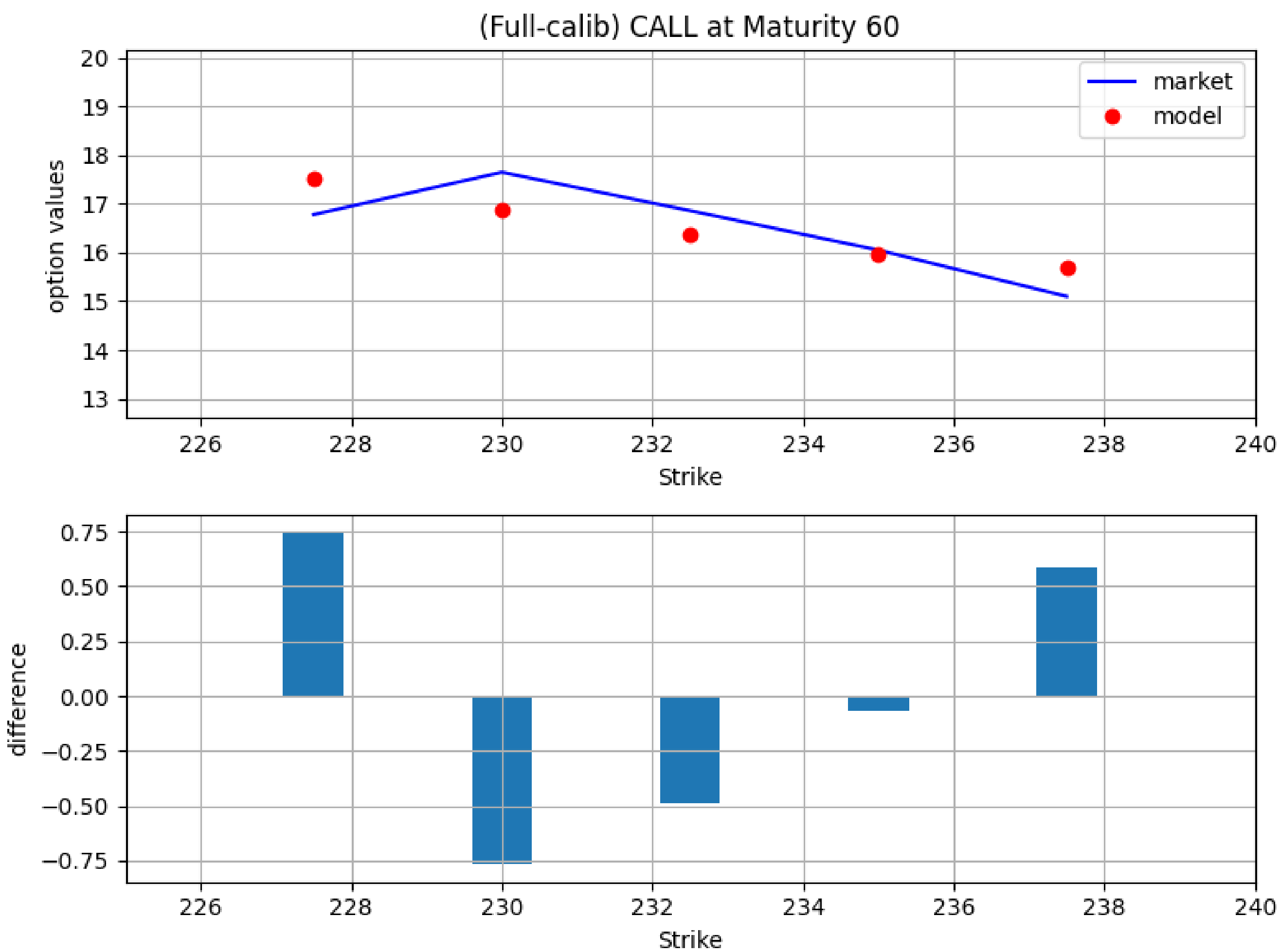
for the CALL options with MSE at 0.349 price mark squared, and

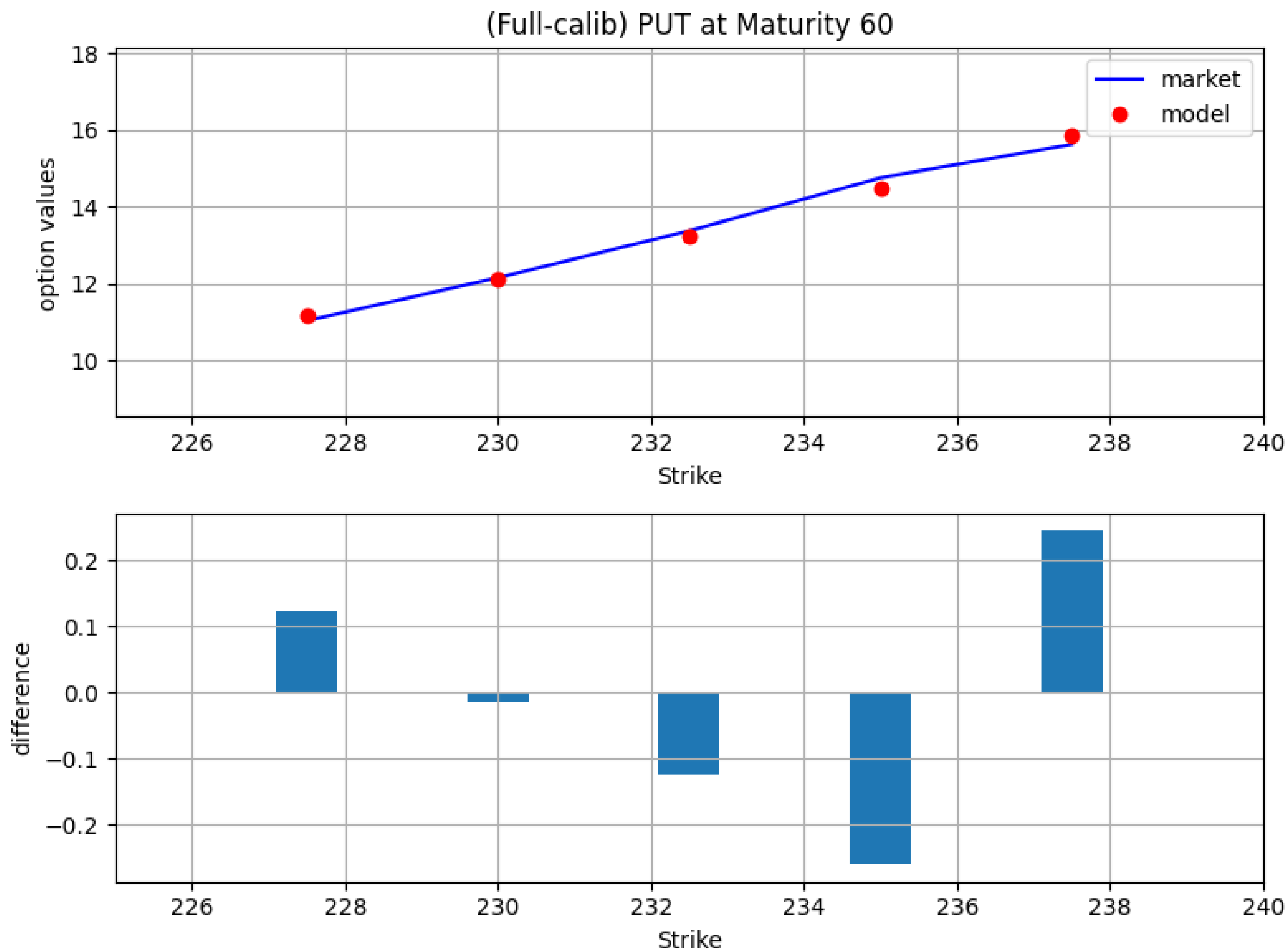
S0	r	lambd	mu	delta	kappa	theta	sigma	rho	v0
232.9	0.015	1.65191	-0.112154	0.396551	2.84644e-08	0.176102	0.000100126	0.70556	0.0242659

for the PUT options with MSE at 0.032 mark.

Carr-Madan based pricing does not compromise with the optimal MSE, in fact the result for the PUT calibration is exactly the same as the Lewis's. It did not shorten the computational time than that of Lewis' considering the amount of optimisation iteration though, which is quite expected as we only deal with a single cash flow product hence the vectorisation property of the FFT doesn't add up any.

Below are the model estimated option price versus the market price.





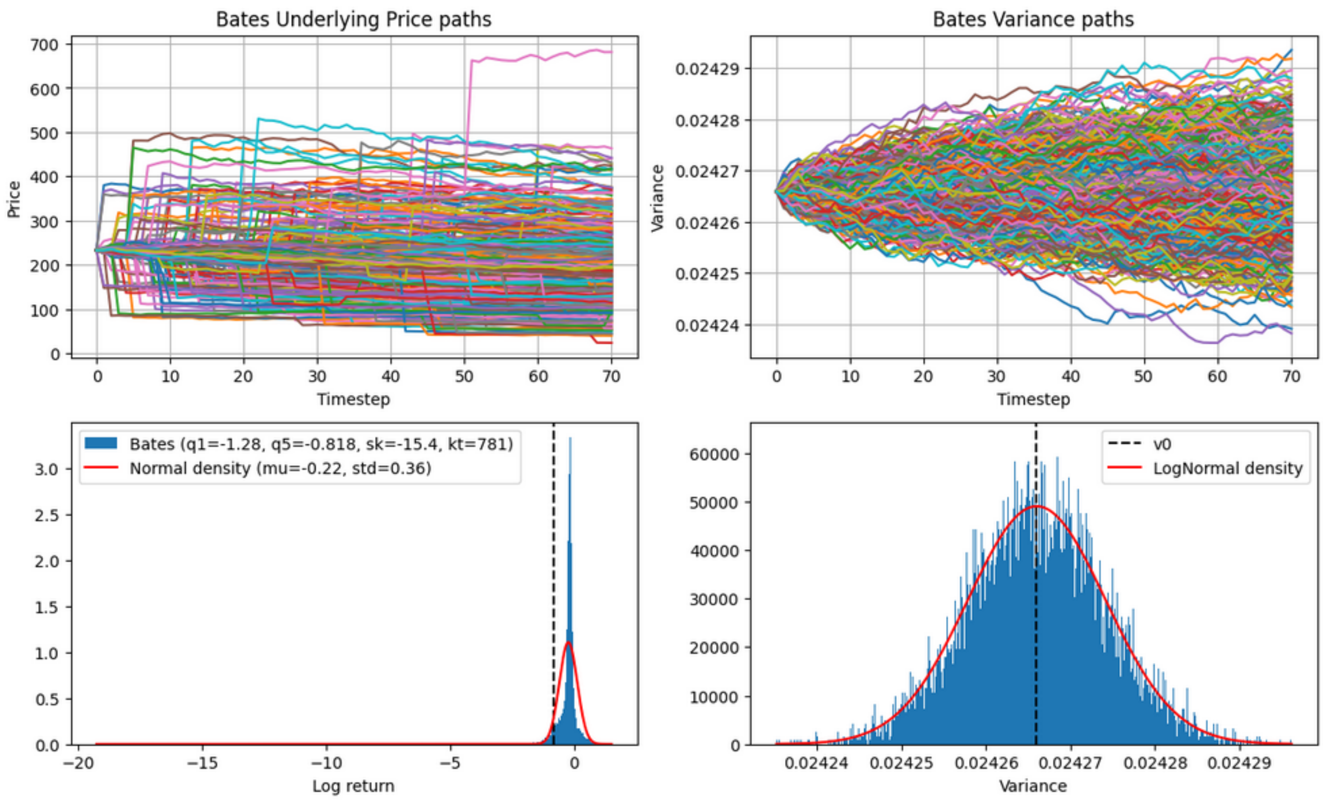
part c: pricing Asian PUT (95% OTM) at 70D to expire with calibrated Bates

Extending the Heston dynamics sampler in section 1.c with the jump diffusion, we sample the Bates dynamics as below:

$$X_{t+\Delta t} = X_t \cdot \exp\left[(r - r_j - 0.5\nu_t)\Delta t + \sqrt{\nu_t}\Delta W_t^{(1)}\right] \cdot (1 + (\exp[\mu + \delta \cdot z_t] - 1) \cdot \Delta J_t)$$
$$\nu_{t+\Delta t} = \nu_t + \kappa \cdot (\theta - \nu_t)dt + \sigma\sqrt{\nu_t}\Delta W_t^{(2)}$$
$$\text{where } d < W^{(1)}, W^{(2)} >_t = \rho \cdot dt,\ z \sim \mathcal{N}(0, 1^2),\ dJ = \mathcal{P}(\lambda dt)$$

Note that we *multiply* “1 + JumpDiffusion” part to the Heston model part instead of adding “JumpDiffusion” term. We found this is the correct derivation that also well explains the fact that the characteristic function of the Bates’ model is the multiplication the Heston and JumpDiffusion process’s.

From the calibrated Bates model (on PUT side) achieved in the part a and b (almost same), we run a monte-carlo simulation of 70 steps (= number of days to expire) over 10000 paths as shown below.



The resultant Asian PUT option price at the 95% moneyness (K=221.255) is 14.49, and the 4% spread marks our offer as 15.07.

The sample paths are largely driven by the jumps on both ways with must bigger size of the jump diffusion than the stochastic volatility diffusion where the final variance stays around the initial variance value. This is what we expect from the calibrated parameters as small κ, σ for volatility barely shifts the variance whilst large jump parameters facilitates the big frequent jumps yet the distribution of the jump size is balanced to open for both signs, ended up having fat tails around zero log return.

Step3

part a. Euribor term structure by calibrated CIR short-rate process based on cubic-spline forward rate curve

The CIR short-rate process is defined as below SDE form by which we can sample with Euler-Maruyama method:

$$dX_t = \kappa(\theta - X_t)dt + \sigma dW_t$$

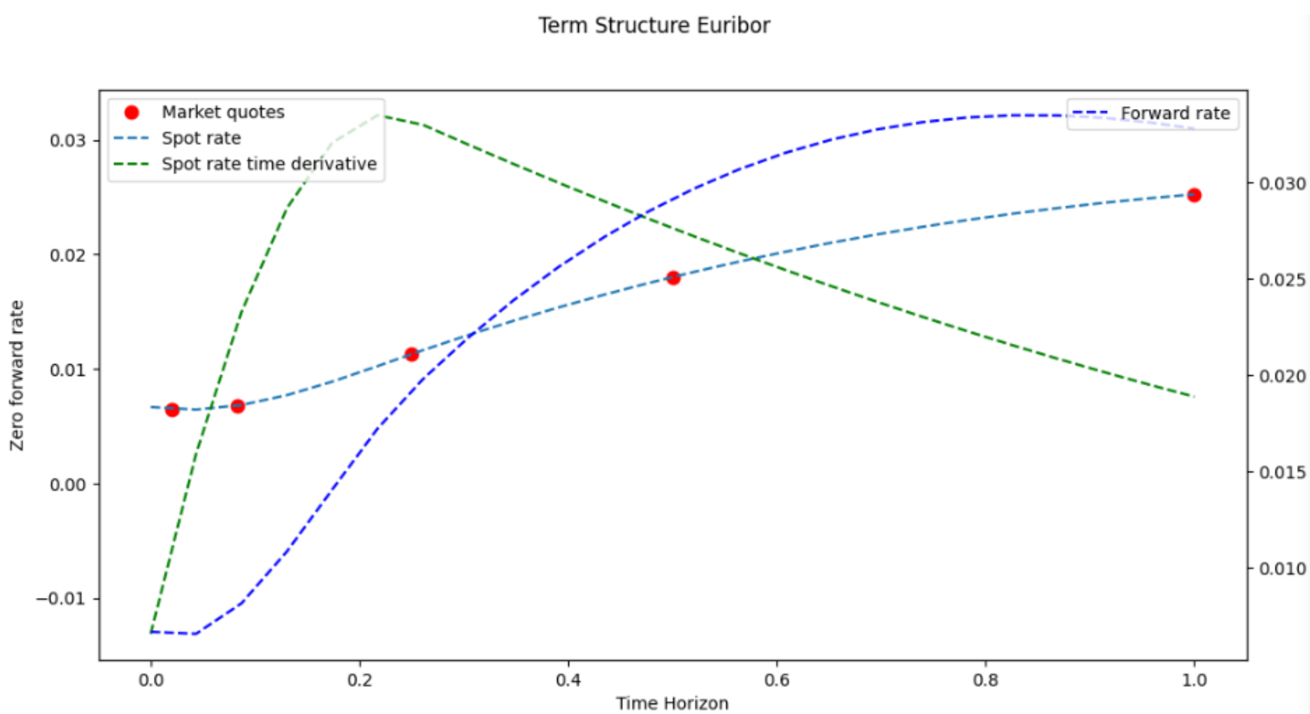
Considering the CIR dynamics as the short-rate process of the forward rate - given that the Vasicek defines the short-rate as $B_t = B_0 \exp(\int_0^t X_t dt)$ - the calibration procedure of the dynamics is as follows:

- Prepare the yield-to-maturity data from the rate points data $c_t = \log(1 + t * r)/t$
- Fit the differentiable non/semi parametric curve estimation model $c_t \sim M(t)$ such as the cubic spline or the Nelson-Siegel-Svenson model
- Convert the rates into the forward rates data via $f_T = M(t = T) + T \cdot \frac{dM(t)}{dt}|_{t=T}$

- Set the initial term-structure point r_0 which typically is set with the overnight funding rate: when no data is available one can get from the spline curve's estimation, although in WQU instruction any nearest tenor was chosen.
- With the estimated initial forward rate f_0 over the tenor space, calibrate the CIR parameters κ, θ, σ:
    1. Calculate the forward rate at t given the current CIR parameters: $f_t = \kappa\theta \frac{\exp[g \cdot t]-1}{2g+(\kappa+g)\exp[gt]-1} + r_0 \frac{4g^2\exp[gt]}{2g+(\kappa+g)(\exp[gt]-1)^2}$ where $g = \sqrt{\kappa^2 + 2\sigma^2}$
    2. Measure the `MSE` between the CIR expected rates and the forward rates from the data
    3. Apply the `MSE` minimiser that searches the CIR parameter values with the MSE of the objective and iterates the above steps towards convergence or stopping criterion. We use the Nelder-Mead simplex downhill method as the optimiser for this practice.

We are given the Euribor marks as below:

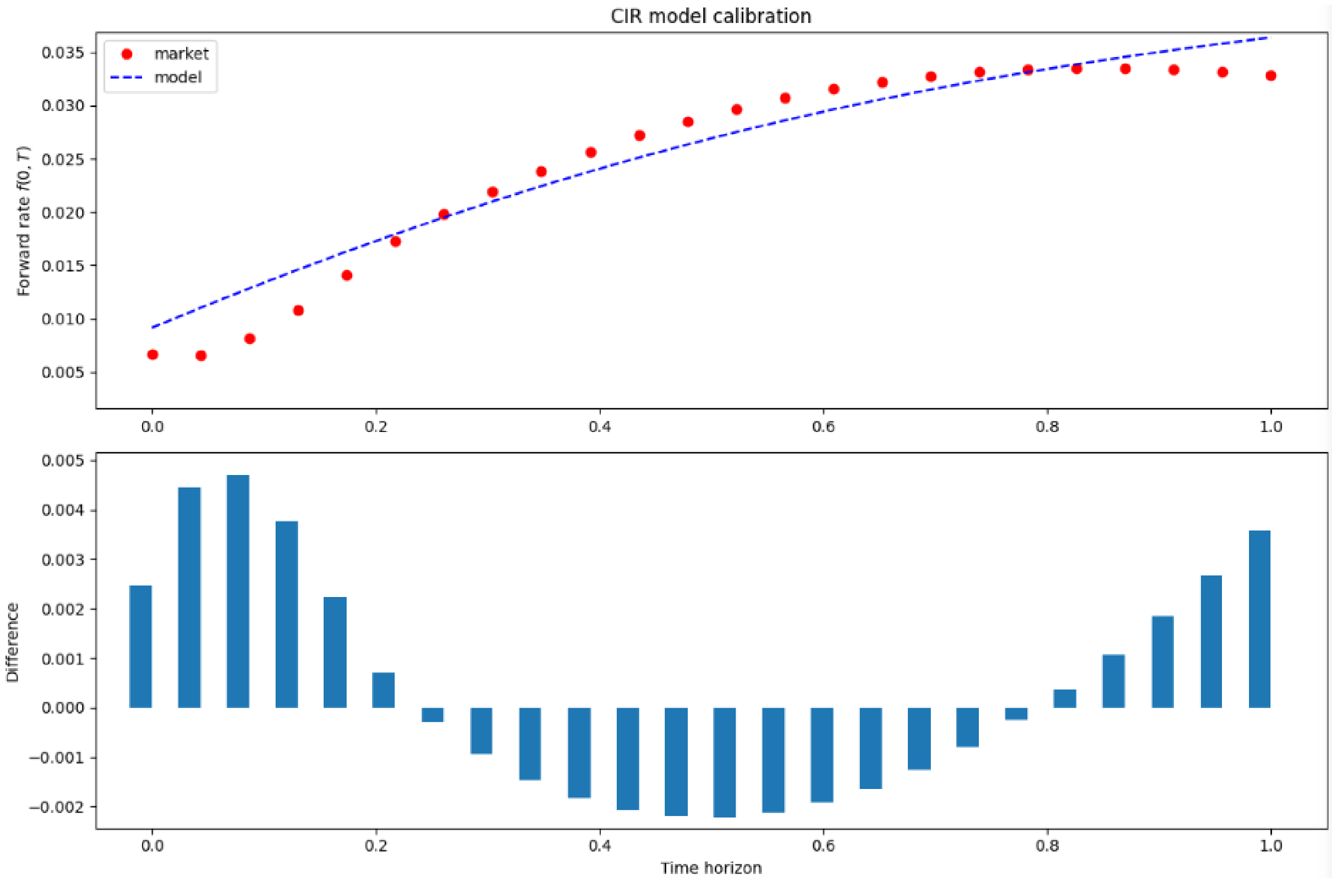
Maturity	r
0.02	0.00648
0.0833333	0.00679
0.25	0.0113
0.5	0.01809
1	0.02556



and the calibrate parameters set becomes:

x0	kappa_cir	theta_cir	sigma_cir
0.00663015	0.68804	0.109374	0.00100031

with the MSE 5.26e-6.

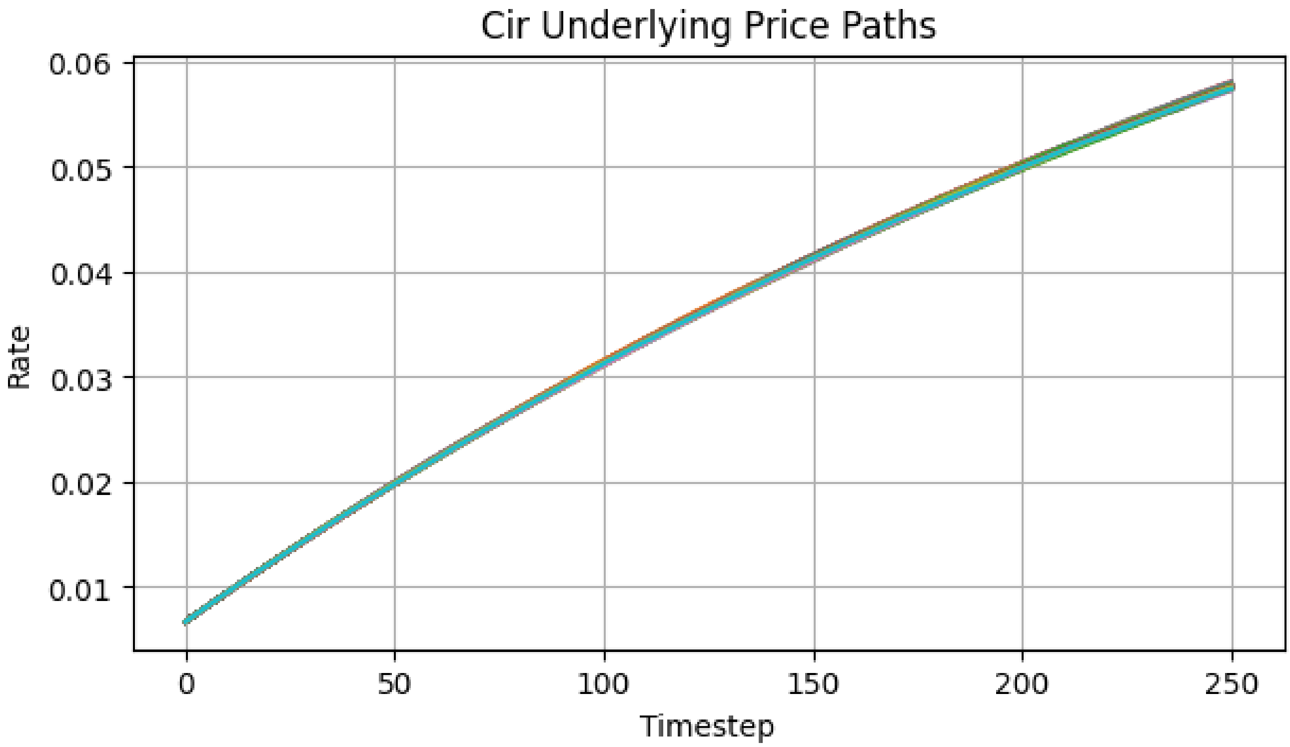


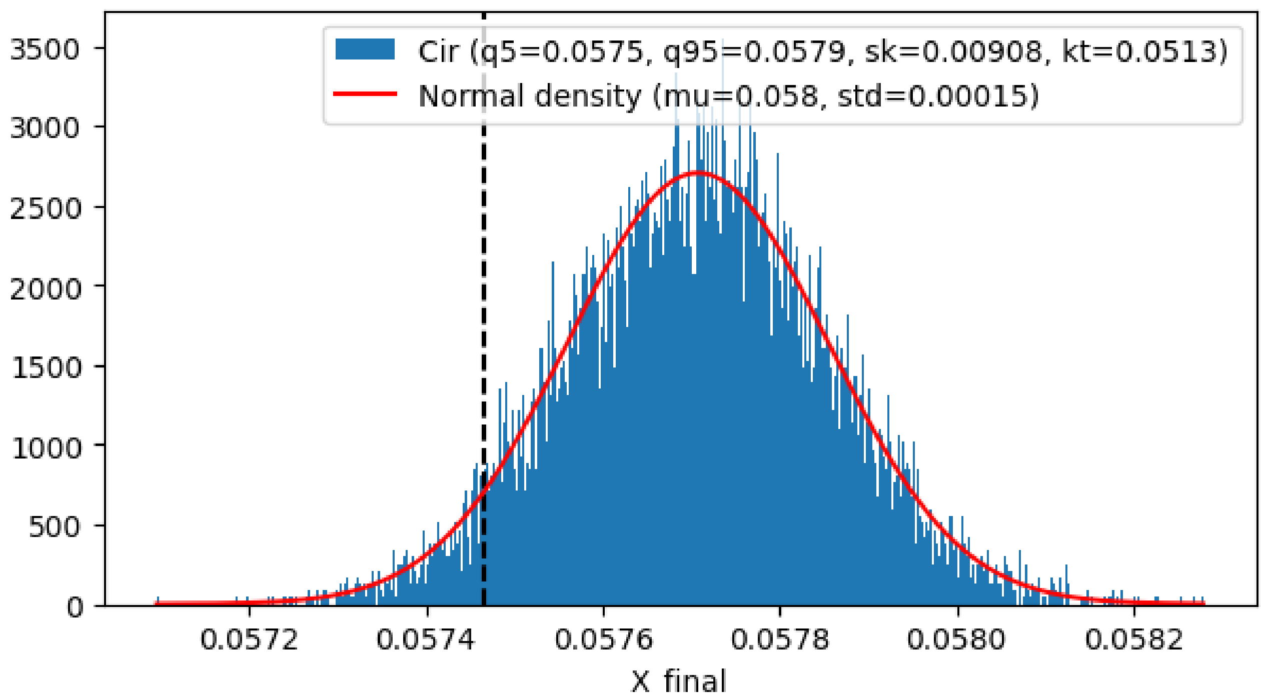
The CIR overestimates the front end, underestimates the middle end, and overestimates the far end. This is a well known limitation of the CIR model for the term structure, which is often exploited by the butterfly trades targeting on less sophisticated market makers.

part b. Sample CIR short-rate process for 12 months and Option pricing on BCC dynamics

i, ii) Expected rate and the confidence bound at 1-year

Below is the 10000 paths sampled result from the calibrated CIR short-rate dynamics in the part a.





Although small and retained, the short-rate varies more on the farther tenor. The final forward rate distribution at 12 months shapes in normal distribution as indicated by the SDE.

5% to 95% percentile confidence bound is found to be (5.75%, 5.79%) at 12-months tenor. The expected value, which is the average of the paths, is at 5.8%.

iii) Impact of the term structure model on the option pricing output at 1-year expiry

In order to understand the impact of the term-structure aware underlying dynamics and its impact on the resultant option price, we focus on the Bates and the BCC dynamics on the 1 year European and Asian put options evaluation via Monte-Carlo sampling methods.

Extending the Bates model with the term-structure on the risk-free rate - CIR model is chosen as the short-rate dynamics in this study:

$$
\begin{aligned}
X_{t+\Delta t} &= X_t \cdot \exp\left[(r_t - r_j - 0.5\nu_t)\Delta t + \sqrt{\nu_t}\Delta W_t^{(1)}\right] \cdot (1 + (\exp[\mu + \delta \cdot z_t] - 1) \cdot \Delta J_t) \\
r_{t+\Delta t} &= r_t + \kappa \cdot (\theta_r - r_t)dt + \sigma_r \sqrt{r_t}\Delta W^{(r)} \\
\nu_{t+\Delta t} &= \nu_t + \kappa \cdot (\theta_\nu - \nu_t)dt + \sigma_\nu \sqrt{\nu_t}\Delta W_t^{(2)} \\
&\text{where } d < W^{(1)}, W^{(2)} >_t = \rho \cdot dt,\ z \sim \mathcal{N}(0, 1^2),\ dJ = \mathcal{P}(\lambda dt)
\end{aligned}
$$

The Fourier-Transformation-Inversion based calibration of the BCC model is similar to that of the Bates, except that the rate is given from the term-structure model at the expiry date. The monte-carlo sampling is also similar to that of the Bates', except that the rate per each timestep is given from the CIR short-rate dynamics.

The Bates dynamics with the fixed risk-free rate from the overnight rate under assumption that the overnight rate keeps at the same level for the next 1-year so that we can carry at the same fuding rate.

On the other hand, the BCC dynamics are with the *dynamic* yield-to-maturity throughout the time preriod of sampling that comes from the calibrated CIR dynamics (in Step3-partA). This means in typical condition, the funding rate is higher than that of the Bates' and that is the case in this study as well.

To generalise over the maturity as to sample farther side, we calibrate the BCC dynamics parameters with market price data on 120D maturity date within 98% - 102% moneyness as shown below:

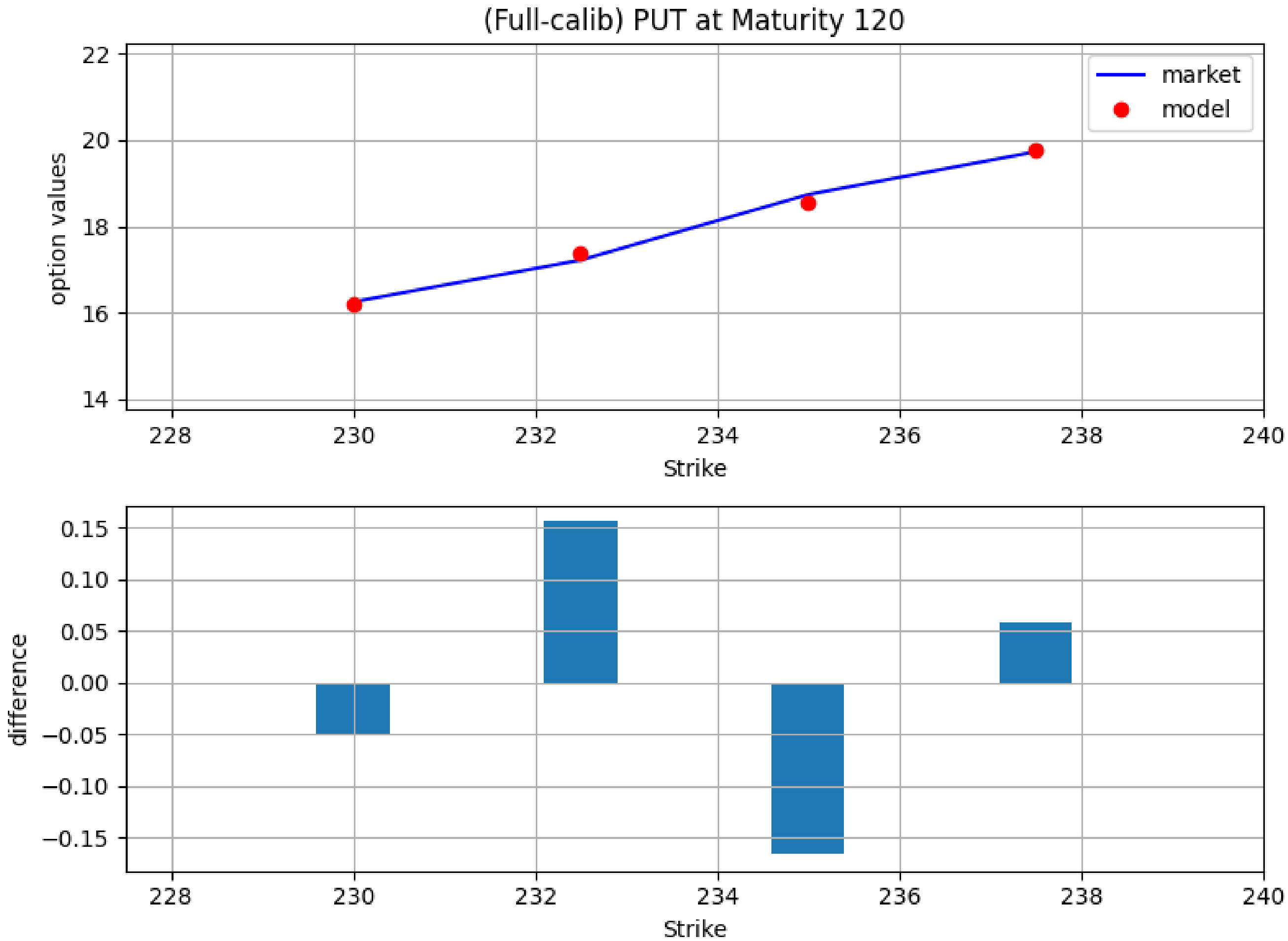
DaysToMaturity	Strike	CALL	PUT	T	r
120	230	24.12	16.25	0.48	0.021873
120	232.5	22.97	17.22	0.48	0.021873
120	235	21.75	18.74	0.48	0.021873
120	237.5	18.06	19.73	0.48	0.021873

which gives

x0	r	lambd_merton	mu_merton	delta_merton	kappa_heston
232.9	0.00661528	1.22788	-0.123941	0.365767	0.00059762

theta_heston	sigma_heston	rho_heston	v0_heston	kappa_cir	theta_cir	sigma_cir
0.153026	0.0135242	-0.012751	0.0252505	0.697536	0.106926	0.00100215

for the PUT side dynamics with MSE 0.016.



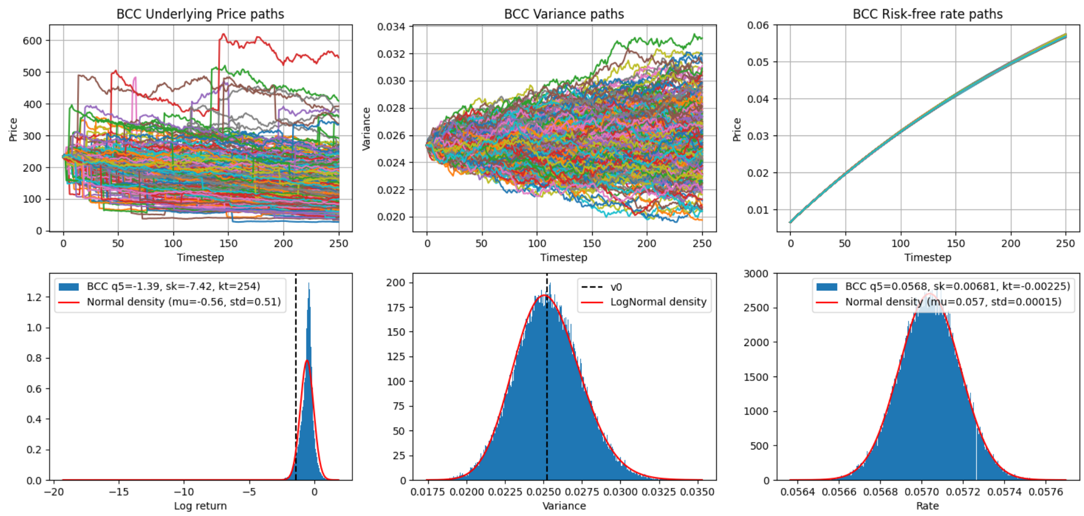
This level of MSE from this BCC model is promisingly low.

The 1-year option price in the same Lewis method to the Bates' in as shown below:

otype	oside	moneyness	calculator	price
european	put	95%	Bates	102.9342
european	put	95%	BCC	79.9749
asian	put	95%	Bates	53.9532
asian	put	95%	BCC	40.3559

, which are much lower than that of the Bates' returns discussed earlier. We suspect it is due to the higher level of the risk-free rate we apply for the BCC model than the Bates model's for why we provide the overnight rate for the all sampling timesteps. Although the direct comparison is not quantitatively meaningful (as their payoff distribution must be different), but what is certain is that the BCC discounts much more.

For sake of the Asian PUT 1Y pricing as well as qualitative observation of the paths, we sampled 10000 paths with the calibrated parameters from the 120D near ATM price points, as metioned above. The fair price from the sample mean is 49.572 from below paths:



This result supports the original postulation that the option price sampled from the BCC with up-slope temr-structure will be lower than that of the Bates - term-structure agonistic risk-free rate - with overnight rate is given as the risk-free rate. One may get the different relationship when the risk-free rate is given as the 1-year yield instead of the overnight, but the jury for the justful rate is still out anyway for selecting a single risk-free rate over the maturity space.