Modeling non-stationarity and finding an equilibrium (Doo Re Song)

Definition

Stationarity of a time-series $\{X_t\}$ is defined as [2-1]:

- Temporal shift-invariance in joint distribution of observations in strong form as $P(\{X_t\}_{t\in[t_0,\dots,t_0+l]}) = P(\{X_{t+h}\}_{t\in[t_0,\dots,t_0+l]})\ \ \forall h, t, t_0, l$
- Mean and autocovariance independence time t and for each lag h in weak form as $\mathbb{E}[X_t] \perp t$ and $Cov[X_t, X_{t+h}] \perp t, h$

The **non-stationary timeseries** is a temporal sequence of observations where the above definition does not hold. Withe the empirical series of observations, we are able to address the statistical significance about whether the time-series is stationary or not via the family of unit root tests such as (Augmented) Dickey Fuller test (ADF) and Kwiatkowski-Phillips-Schmidt-Shin test (KPSS) - see [2-1] or previous GWP for further details. As the stationarity assumption is required for applying AR(I)MA and (G)ARCH, non-stationarity is a trouble to the real-world data modeling. Further venues on modelling a single non-stationary time-series includes: inspecting possible integrated stationarity of higher orders, or applying model with hidden state underlying to observations and attempting to extract stationarity in residuals - such as GARCH on Kalman Filter [2-2].

Another subject of analysis is to explain a time-series by another times-series for discover the cross-sectional relationship. Consider two non-stationary time-series $\{X_t\}, \{Y_t\}$. **Cointegration** of the two time-series is defined as the existence of a linear combination of the two - such as $y_t - (\beta_0 + \beta_1 x_t) = \epsilon_t$ - being stationary. Concretely, we determine that the two time-series are cointegrated if the residual term ϵ_t is $I(0)$ stationary, if the two are $I(1)$. The counter hypothesis of the cointegration is the **spurious regression** condition where the residual is not $I(0)$ stationary but is in $MA(q > 0)$ process indicating that their linear relationship is indeed made of accumulation of random noises, where the Durbin-Watson test statistic should provide number distant from 2 meaning the noise is autocorrelated.

One framework of the cointegration test towards modeling in this subject is Engle-Granger two-step procedure. For the above example, we check the $I(1)$ stationarity of the two time-series, then perform ordinary linear regression for one to another to get the residual sequence, then finally we apply Phillips-Ouliaris test to check the existence of unit root of the residual chain as H_{null}, which means the sequence is cointegrated if is rejected. Under the cointegration assumed, we can model with the Error Correction Model for one time-series by another as [2-3] $\Delta y_t = c + \sum_{i=0}^{p} \alpha_i \Delta x_{t-i} - \lambda \epsilon_t + u_t$. The residual term u_t should offer near zero autocorrelation over any lag to support its $I(0)$ stationarity. The error term $\epsilon_t := y_t - (\beta_0 + \beta_1 x_t)$ correction applied with the error decay coefficient $\lambda \in [0, 1]$ governs the short-term convergence of the combined time-series.

Extending the framework to more multi-variates that regresses to each by all, one can model with the Vector Error Correction Model (VECM(p)) as

$$\Delta \underline{O_t} = \underline{C} + \sum_{i=1}^{p} \underline{\underline{\Gamma_i}} \cdot \Delta \underline{O_{t-i}} + \underline{\underline{\Pi}} \cdot \underline{O_{t-1}} + \underline{u_t}$$

where the error decay matrix is Π. The matrix is the determinant of stationarity of which its empirical rank from Johansen-Cointegration test provides the statistical confidence on the cointegration property of the bundled time-series at $I(0)$. Finally, the long-run **equilibrium** is a linear combination where the i-th cointegral (the error term) $o_t^{(i)} - \sum_{j\neq i} \beta_{j,t} o_t^{(j)}$ is $I(0)$ stationary [2-4], which can be confirmed by usual unit root tests such as ADF. Ergodicity supports the uniqueness of the cointegral and independence of initial conditions.

Description

Non-stationarity of time-series invalidates the application of well known time-series models such as ARMA or ARCH, however some linear combinations of the time-series bundles may produce stationary time-series that is mean-reverting on the long-term equilibrium.

Demonstration

We investigate the conintegrated property of three major Emerging Market (EM) currencies value against the US Dollar: INRUSD, IDRUSD, and BRLUSD. Those currencies represent (India, SE Asia, Latam) highly clustered volatility with empirical integrated order higher than 1 as their economic strength is self-evolving due to their size, yet we suspect any form of cointegrated movement may exist value as the EMs are heavily influence of their trade performance against the US.

With their daily close mark for a decade, this demo provides the empirical evidence practice consists of three parts:

1. Examination of non-stationarity of each time-series
2. Measurement of cross-(auto)correlation among them as navigate effective vectorised estimator
3. Derivation of the cointegrated model with proof of existence of equilibrium

In `Diagram` section, key steps for this analysis are described by plots describing result per step. In `Diagnosis` section, the takeaway for each step is explain that validates (or invalidate) subsequent step of analysis towards the statistically significant equlibrium modeling from multiple EM currencies' spot FX rates. For modeling and diagnosis, we use year 2015 to 2024 then validates with 2025/Q1.

Diagram

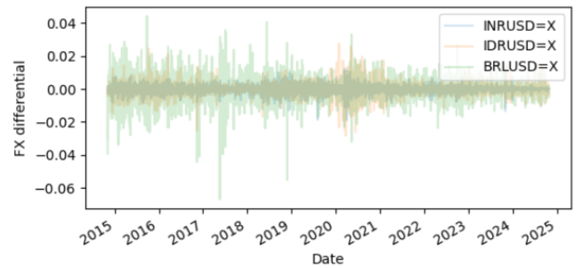
Note,

- FX spot rates here are scaled to align around 1 for plotting purposes: INRUSD's scaled by 50, IDRUSD's 10000, BRLUSD's 3.
- Read each figure from left to right, top to bottom.

Figure 1 explains the single timeseries analysis for their non-stationarity:

- 1-1: Timeseries chart of FX spot level
- 1-2: Timeseries chart of FX spot differentials
- 1-3: Augmented Dickey-Fuller test results by p-value per currencypair (column) for level or differential FX rate (row)
- 1-4: Autocorrelation plot for each currency pair

Figure 1: single timeseries assessment	Timeseries charts and stationarity checks



	INRUSD=X	IDRUSD=X	BRLUSD=X
I(0)_ADF	1.168386e-01	1.312913e-04	0.133379
I(0)_KPSS	4.022219e-04	2.051639e-02	0.000100
I(1)_ADF	2.397681e-22	1.869800e-16	0.000000
I(1)_KPSS	8.209358e-01	8.784561e-01	0.209198

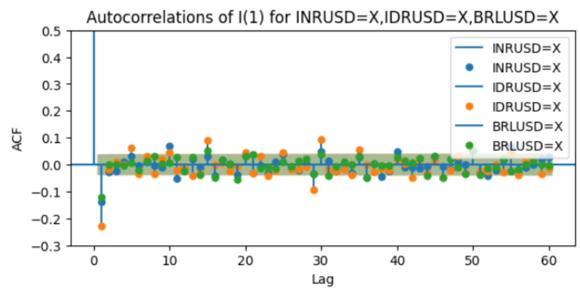


Figure 2 explains the cointegration and equilibrium analysis:

- 2-1: Autoregression(p) order selection and Johansen rank test
- 2-2: Vector ECM calibration result
- 2-3: Cointegration($IDRUSD, INRUSD, BRLUSD$) pastcast
- 2-4: Augmented Dickey-Fullter test on the cointegral

Figure 2: VECM modeling	VECM modeling and Cointegrated timeseries char as well as statistics

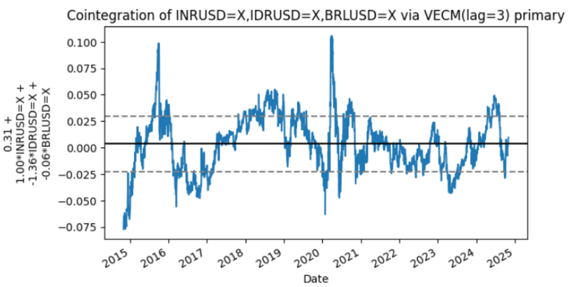
VAR Order Selection (* highlights the minimums)

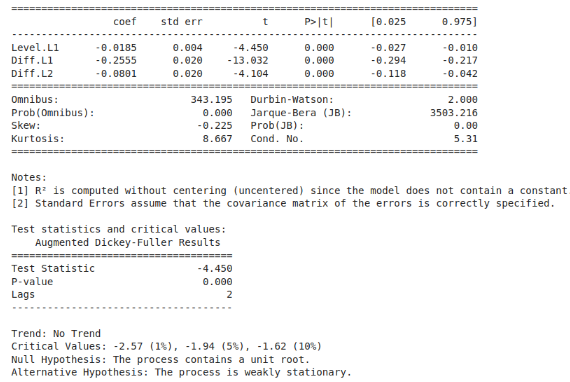
	AIC	BIC	FPE	HQIC
0	-18.59	-18.58	8.477e-09	-18.58
1	-32.61	-32.59	6.860e-15	-32.60
2	-32.72	-32.68*	6.149e-15	-32.71*
3	-32.73*	-32.66	6.109e-15*	-32.70
4	-32.73	-32.64	6.133e-15	-32.69
5	-32.72	-32.61	6.162e-15	-32.68

	Test statistic	Critical values (90%)	Critical values (95%)	Critical values (99%)
rank=0	36.539955	27.0669	29.7961	35.4628
rank<=1	11.609500	13.4294	15.4943	19.9349
rank<=2	2.497733	2.7055	3.8415	6.6349

```
              Loading coefficients (alpha) for equation INRUSD=X
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
ec1           -0.0026      0.002     -1.359      0.174      -0.006       0.001
              Loading coefficients (alpha) for equation IDRUSD=X
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
ec1            0.0125      0.003      3.962      0.000       0.006       0.019
              Loading coefficients (alpha) for equation BRLUSD=X
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
ec1           -0.0043      0.006     -0.721      0.471      -0.016       0.007
          Cointegration relations for loading-coefficients-column 1
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
beta.1         1.0000          0          0      0.000       1.000       1.000
beta.2        -1.3713      0.214     -6.395      0.000      -1.792      -0.951
beta.3        -0.0528      0.057     -0.926      0.355      -0.165       0.059
const          0.3117      0.119      2.631      0.009       0.079       0.544
==============================================================================
```

Figure 2: VECM modeling	VECM modeling and Cointegrated timeseries char as well as statistics





```
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
Level.L1      -0.0185      0.004     -4.450      0.000      -0.027      -0.010
Diff.L1       -0.2555      0.020    -13.032      0.000      -0.294      -0.217
Diff.L2       -0.0801      0.020     -4.104      0.000      -0.118      -0.042
==============================================================================
Omnibus:                      343.195   Durbin-Watson:                   2.000
Prob(Omnibus):                  0.000   Jarque-Bera (JB):             3503.216
Skew:                          -0.225   Prob(JB):                         0.00
Kurtosis:                       8.667   Cond. No.                         5.31
==============================================================================

Notes:
[1] R² is computed without centering (uncentered) since the model does not contain a constant.
[2] Standard Errors assume that the covariance matrix of the errors is correctly specified.

Test statistics and critical values:
    Augmented Dickey-Fuller Results
=====================================
Test Statistic                 -4.450
P-value                         0.000
Lags                                2
-------------------------------------

Trend: No Trend
Critical Values: -2.57 (1%), -1.94 (5%), -1.62 (10%)
Null Hypothesis: The process contains a unit root.
Alternative Hypothesis: The process is weakly stationary.
```

Diagnosis

Figure 1 shows that the three EM currencies non-stationary in level I(0), whilst I(1) is suggested to be stationary via ADF test with 'constant × time' trend adjustment except IDRUSD for which the long-term govermental control on the FX rate is suspected and/or statistically the linear trend adjustment may have absorbed the integration by chance. The ACF plot shows that the first lag is influential although some other lags sit outside of 2 standard errors confidence interval, suggesting a possibility of significant AR beyond lag 1.

Figure 2 starts with probing the optimal AR lag to constitute the VECM model. Whilst the BIC favours in lag 2, the Final Prediction Error (L2 norm of the estimated test error matrix among timeseries) suggest to explore more upto lag 4. Given that we observed the autoregressive nature of the timeseries, we adopt lag 3 at cost of more AR parameters.

The application of the VECM is also validated with the Johansen rank test where the zero-rank hypothesis is rejected and rank one is accepted. The calibration summary of the VECM(p=4) in Figure 2-2 shows that the β of IDRUSD is highly confident to be non-zero, constant is significant beyond two-sigma confidence, and BRLUSD is found to be barely relevant in terms of the same t-statistics p-value which also results in very contribution to the noise model with low β.

The cointegration pastcast chart in Figure 2-3 visualises the **mean-reverting** property the cointegrated timeseries calibrated with the VECM model around near zero. According to the unit root testing results in Figure 2-4, the ADF test statistics on the cointegration suggests that the timeseries is weakly-stationary at AR(2), suggesting the existence of the long-term equilibrium. The Durbin-Watson statistics is at 2.0 which supports this cointegration's noise is not spuriously autocorrelated.

Damage

Primary potential issues stem from the equilibrium model assumptions where we apply the VECM with error-term normality assumed, the generalisation quality of the calibrated model is not validated, and the model parameters prior can change over time.

- High skewness implies the high odds of tail event occurence

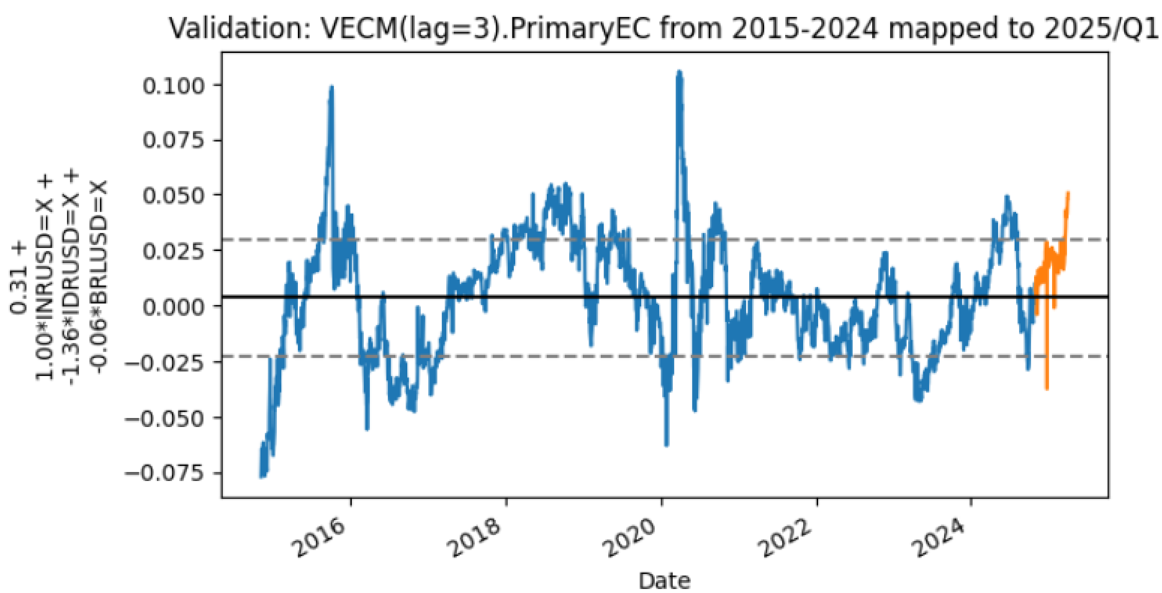
- Whilst pastcast looks promising, the model is not validated with the future data
- Trends can change are often caused by exogenous events, and this can break the whole model

Directions

To overcome the non-normality, one can transform the cointegration values into a standardised score with a probability distribution that handles skewness and kurtosis such as (skewed) T-distribution $T(x)$. For instance, transforming the value with T-distribution's CDF then reverse map into the Z-score - $c_t \rightarrow T_{CDF}(x) \rightarrow \mathcal{N}^{-1}(T_{CDF}(x))$ - will provide (skew-)kurtosis free score that tells the divergence to the equilibrium. This transformation enables broader time-series model candidates such as PCA without underestimation of tail-events. One can also perform volatility clustering measurement and GARCH modeling to estimate the self-excitement property of extreme events.

To validate the mean-reverting property of the cointegrated timeseries, one can split the historical data into training data to calibrate the model and apply the model for forecasting of later data points. The validation test statistics such as BIC of the VECM model will indicate the generalised model quality of cointegration construction with generalised hyperparameters such as max lags. As shown in the below validation, the EM index reverts back from negative to positive and penetrates the 1σ boundary.

The equilbrium will be further tested in terms of future reversion and the profile of this upshoot compared to the previous ones where the ADF and Durbin-Watson tests on residual indicates the validity as we discussed in `Diagnosis` section. Would this model survive at the verge of the trade war? Each country's industrial and governmental profile result in different impact on the trades against the US hence the FX rate. This leads to the next issue.



Lastly, dynamic trend is a well-kown property of financial time-series and a constant trend line or curve for a decade does not fit well, as the economic regime changes. One simple format to account for the dynamic trend is to construct a piece-wise trend function over the temporal space and applied for each timeseries before applying the VECM model. The knots of this piece-wise trend function are the change points detected as the split point where the two time-series splitted by the point shows the change of regime. Regime switching model typically estimates this *transition* in trend or autoregressive properties by state-space model that switches at tuning points, as a result providing wider stationarity conditions than those without regime awareness [2-5].

Deployment

We consider of utilising the cointegration as a tradable index for relative value (RV) strategy.

- The index consists of weighted sum of (scaled) currency spots against US Dollars each of which are tradable.
- Statistical test result shows the long-term equilibrium exists, which means we can point out relative cheap to long and rich to short throughout the temporal space and that divergence from equilibrium will likely recover.

For the model maintanence persepective, we forecast the index by retraining every day and recalibrate to retain the model quality. Recalibration is required as the forecasting horizon is long to observe mean-reversion whilst the index shall re-evaluate every day. If the confidence of the mean-reverting property drops, we weight less on this RV signal on our portfolio of strategies. If the parameters of the model changes drastically, we dissaprove the model as the underlying equilibrium does not persist.

For the optimal trading strategy perspective, we can extract the state condition which give high odds of mean-reversion as well as good profit levels. For instance, the any overshooting beyond 1 standard deviation has provided rapid recovery toward the equilibrium point as shown in the Figure 2-3, hence we long/short when the index reaches $-1\sigma/+1\sigma$ with out-of-the-money put/call options to cover tails events risk.