Reference

1. An Undergraduate Introduction to Financial Mathematics. J. Robert Buchanan. 2010.
2. Mathematical Models of Financial Derivative, Chapter5. Yue Kuen Kwok. 2008.
3. American Options relation between greeks. Quantitative Finance StackExchange

Tasks

Problems

Call-Put Parity in the context of binomial tree model

- Q1. Does put-call parity apply for European options? Why or why not?

- Q2. Rewrite put-call parity to solve for the call price in terms of everything else.

- Q3. Rewrite put-call parity to solve for the put price in terms of everything else.

- Q4. Does put-call parity apply for American options? Why or why not?

Binomial tree Use the following values and parameters:
$S_0 = 100$, $r = 5\%$, $\sigma = 20\%$, $T = 3$ months

- Q5. Price an ATM European call and put using a **binomial tree**:

  a. Choose the number of steps in the tree you see convenient to achieve reliable estimates.

  b. Briefly describe the overall process, as well as a reason why you choose that number of steps in the tree.

- Q6. Compute the Greek Delta for the European call and European put at time 0:

  a. How do they compare?

  b. Comment briefly on the differences and signs of Delta for both options. What does delta proxy for? Why does it make sense to obtain a positive/negative delta for each option?

- Q7. Delta measures one sensitivity of the option price. But there are other important sensitivities we will look at throughout the course. An important one is the sensitivity of the option price to the underlying volatility (**vega**):
  - a. Compute the sensitivity of previous put and call option prices to a 5% increase in volatility (from 20% to 25%). How do prices change with respect to the change in volatility?
  - b. Comment on the potential differential impact of this change for call and put options.

American option

- Q8. Repeat Q5, but this time consider options (call and put) of American style. (Answer sections a and b of Q5 as well).
- Q9. Repeat Q6, but considering American-style options. (Answer/comment on sections a and b of Q6 as well).
- Q10. Repeat Q7, but considering American-style options. (Answer/comment on sections a and b of Q7 as well).
- Q11. If the team answered Q1 as "Yes" (i.e. that put-call parity holds), then show that the European call and put satisfy put-call parity. Comment on the reasons why/why not the parity holds, as well as potential motives.

- Q12. If the team answered Q4 as "Yes" (i.e. that put-call parity holds), then show that the American call and put satisfy put-call parity. Comment on the reasons why/why not the parity holds, as well as potential motives.

- Q13. Confirm that the European call is less than or equal to the American call. Show the difference if any and comment on the reasons for this difference, would this always be the case?

- Q14. Confirm that the European put is less than or equal to the American put. Show the difference if any and comment on the reasons for this difference. For example, would this always be the case?

Trinomial tree

- Q15. Select 5 strike prices so that Call options are: Deep OTM, OTM, ATM, ITM, and Deep ITM. (E.g., you can do this by selecting moneyness of 90%, 95%, ATM, 105%, 110%; where moneyness is measured as K/S_0):

  a. Using the trinomial tree, price the Call option corresponding to the 5 different strikes selected.

  b. Comment on the trend you observe (e.g., increasing/decreasing in moneyness) in option prices and whether it makes sense.

- Q16. Repeat Q15 for 5 different strikes for Put options. (Make sure you also answer sections a and b of Q15).
- Q17. Repeat Q15, but this time consider Call options of American style. (Answer sections a and b of Q15 as well).

- Q18. Repeat Q16, but this time consider Put options of American style. (Answer sections a and b of Q15 as well).
- Q19. **Graph #1**. Graph **European call prices** and **put prices** versus stock prices.

- Q20. **Graph #2**. Graph **American call prices** and **put prices** versus stock prices.

- Q21. **Graph #3**. Graph **European and American call prices** versus strike.

- Q22. **Graph #4**. Graph **European and American put prices** versus strike.
- Q23. For the 5 strikes that your group member computed in Q15 and Q16, check whether **put-call parity holds** (within sensible rounding). Briefly comment on the reasons why/why not this is the case.
- Q24. For the 5 strikes that your group member computed in Q17 and Q18, check whether **put-call parity holds** (within sensible rounding). Briefly comment on the reasons why/why not this is the case.

Dynamic Delta Hedging

- Q25. Use the following data: $S_0 = 180$, $r = 2\%$, $\sigma = 25\%$, $T = 6$ months, $K = 182$:
  - a. Price a European Put option with the previous characteristics using a 3-step binomial tree (you do not need code for this).

  - b. Pick one path in the tree.
    * i. Describe the Delta hedging process (how many units of the underlying you buy/sell, ...) of that path throughout each step if you act as the **seller of the Put option**.

* ii. Make sure you include a table with how your cash account varies at each step (you can follow the format in the slides from Lesson 3 in Module 1). Also, assume you can buy fractions of the underlying asset shares.

- Q26. **Using the same data from Q25, price an American Put option.** Still, assume you are **acting as the seller of this put**. Consider now 25 steps in the tree (do this via python code).
  a. Compute the delta hedging needed at each node in each step.
  b. Show the evolution of the cash-account throughout the different steps for one path of your choice.
  c. Comment on the Delta hedging process as compared to the European option case.
- Q27. **Finally, repeat Q26 considering now an Asian ATM Put option.** Comment on your results as compared to the regular American Put option case of Q25.

Assume no stock dividend throughout this report unless specified otherwise per question.

Call-Put parity of options

Q1. Yes, for European-style options on a non-dividend-paying stock, the put–call parity $c_0 + Ke^{-rT} = p_0 + S_0$ holds exactly. We can construct a unique replication of the European option via the underlying asset with zero-coupon bond at yield r for every state transition in time and price level.

Q2. $c_0 = S_0 + p_0 - Ke^{-rT}$

Q3. $p_0 = c_0 - S_0 + Ke^{-rT}$

Q4. No, as the American option has early excercise optionality. American calls on a non-dividend stock never get exercised early, so call parity still holds there, but American puts can be worth more than that of the European puts due to the extra early-exercise value. Instead, we have an inequality from the no-arbitrage principle $c_0^{(A)} + Ke^{-rT} \leq p_0^{(A)} + S_0$ [1] as otherwise the portfolio with long-put/stock and short-call/cash has cash-flow at inception $c_0^{(A)} + Ke^{-rT} - (p_0^{(A)} + S_0)$ positive while negative future payoffs never exist.

Price and Sensitivities of options

Tree models for option pricing and vega

Q5-a. 4.61 for call with N=2160, 3.37 for put with N=2160

Q5-b. Construct binomial tree of Stock price transition with fixed up/down price mutiplier and a fixed risk-neutral state transition probability. Backward induce the option price from the terminal time (at expiry) condition as the option payoff and back-transit price as the weighted average of the future prices. I chose the list of N as $[3, 12, 60, 480, 2160]$ to observe the price development at monthly, weekly, biz-daily, market-hourly, and hourly where the hourly dynamics is fine enough to expect rough convergence (absoulte tolerance < 0.01) and monthly is too coarse to observe significant numerical deviation.

Q6-a. 0.57 for call and -0.43 for put with N=2160. The delta for call is positive and is negative for put. The difference of call delta and put delta is one, as the partial derivative of the call-put parity $\frac{\partial(c-p)}{\partial S} = \partial(S - Ke^{-rT})/\partial S = 1$ for the European option.

Q6-b. Positivity of the call's delta stems from the positivity of the payoff at expiry and the call price is a discounted expected value of paths, hence the price sensitivity to the underlying's price is positive. Negativity of the put's delta is explained the same as the payoff at expiry is negative. The delta of option price the option price first order partial derivative of the underlying's price and its absolute value also means the odd of the option is at or in-the-money at expiry (in forward measure) according to the definition of delta in Black-Scholes formula.

Q7-a. 0.20 for call and 0.20 for put as option price change per 1% change of volatility with N=2160. Increase in volatility raises the price of both call and put long position for European option.

Q7-b. Without the change of the underlying price nor risk-free rate nor time-to-expiry, change in volatility results in correlated change of option price. Positivity of vega for both call and option also allow pure volatility trading with holding horizon before expiry by which the value of straddle can increase when volatility increases in short amount of time even when the underlying price reverts back and time-value decay is less than the vega-contributed increment.

Q8-a. 4.61 for call and 3.48 for put with N=2160

Q8-b. Overall pricing simulation is the same of the European option pricing routine except that the option can be excercise each step hence the price is not just the discounted weighted-average of the next option prices but the max between that and the immediate payoff. The call price does not change as the optimal stopping time is at expiry, whilst the put price is higher than that of PUT as the early excercise increases the chance of non-zero payoff due to the existence of the positive early-exercise region [2].

Q9-a. 0.57 for call and -0.45 for put with N=2160. The delta for call is positive and is negative for put. The difference of call delta and put is bigger than one, as is stated in the call-put parity inequality for the American option, in other words the sensitivity on underlying's price of the American put option increased on its magnitude.

Q9-b. The statement is almost the same as that of Q6-b. Note, the delta of the American put does not represent the change of being exercised anymore as the early exercise means non-existence of single forward measure that is martingale.

Q10-a. 0.20 for call (same as the European call's) and 0.20 for put (sightly less than that of European put's) as option price change per 1% change of volatility with N=2160. Increase in volatility raises the price of both call and put long position for American option.

Q10-b. The statement is almost the same as that of Q7-b. Note though, the shape of vega for the American put is not symmetric around in-the-money as higher volatility for out-of-money facilitates further chances of early exercise making a steep decline of the vega on the out-of-money region [3]

Q11. Verification of put-call parity for European option with Team A's results:

Call price (C) = 4.61, Put price (P) = 3.37 Given: Stock price (S) = 100, Strike price (K) = 100, Risk-free rate (r) = 0.05, Time to maturity (T) = 0.25 Calculation:

- $C + Ke^{-rT}= 4.61 + 100 * e^{-0.05*0.25}= 103.37 $
- $P + S = 3.37 + 100 = 103.36 $

Conclusion: Within the rounding error, put-call parity holds for European options. Put-call parity remains valid due to the fundamental assumption of no arbitrage in options pricing. When this parity is disrupted, arbitrage opportunities arise, enabling traders to secure immediate, risk-free profits by structuring an appropriate portfolio. However, temporary deviations can occur, even for European options, due to market frictions, dividend effects, or occasional mispricing.

Q12. Not applicable as answer to Question Q4 is false., Q13. Results: European call = 4.61, American call = 4.61. This confirms that an American call is not cheaper than a European call. Since early exercise is not optimal for a call option, the American call price is effectively the same as the European call price. Since early exercise is not optimal for a call option, the American call price is effectively the same as the European call price.,

Q14. Results: European put = 3.37, American put = 3.48. This confirms that an American put is not priced lower than a European put. The 0.11 price difference in this scenario arises from the flexibility to exercise the American put early. However, if early exercise does not provide an advantage - such as the dynamics is with zero risk-free rate -, this difference may not be present.

Moneyness and price

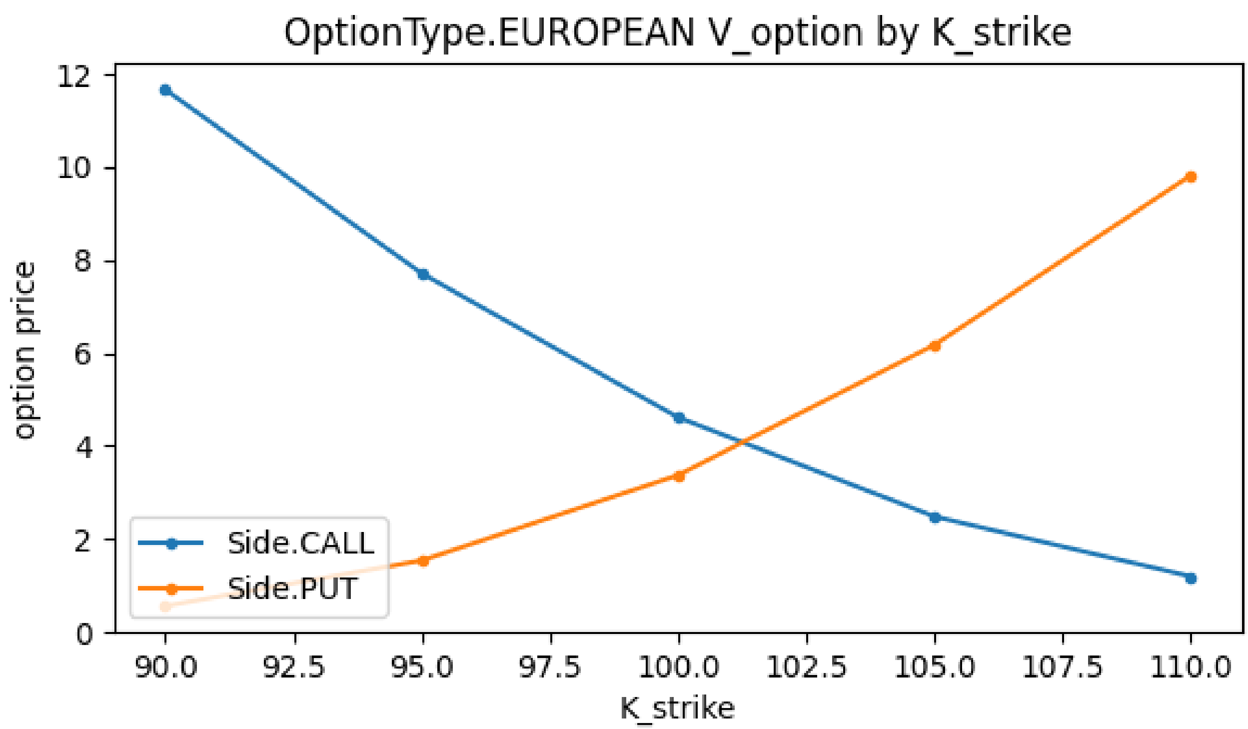
Q15,16-a. $[11.67, 7.71, 4.61, 2.48, 1.19]$ for European call and $[0.55, 1.53, 3.37, 6.17, 9.84]$ for European put. Note, we did not change the order of moneyness for the put option hence but in ascending order of the strike price hence the put option price order is in opposite order of the call's.

Q15,16-b. Both call and put increases its value from OTM to ITM where the moneyness is in ascending order for the call and in descending order for the put. OTM offers less chance to strike with positive option than ITM option, hence the backward discounted weighted sum decreases when the stike goes far out from the current underlying.

Q17,18-a. $[11.67, 7.71, 4.61, 2.48, 1.19]$ for American call and $[0.56, 1.57, 3.48, 6.42, 10.33]$ for American put. As expected, the American put price is higher than the European put's at the same moneyness due to the early-exercise optionality premium.

Q17,18-b. The trend is the same as what we observed with the European options except with stiffer increment on the American put in line with the moneyness.

Q19.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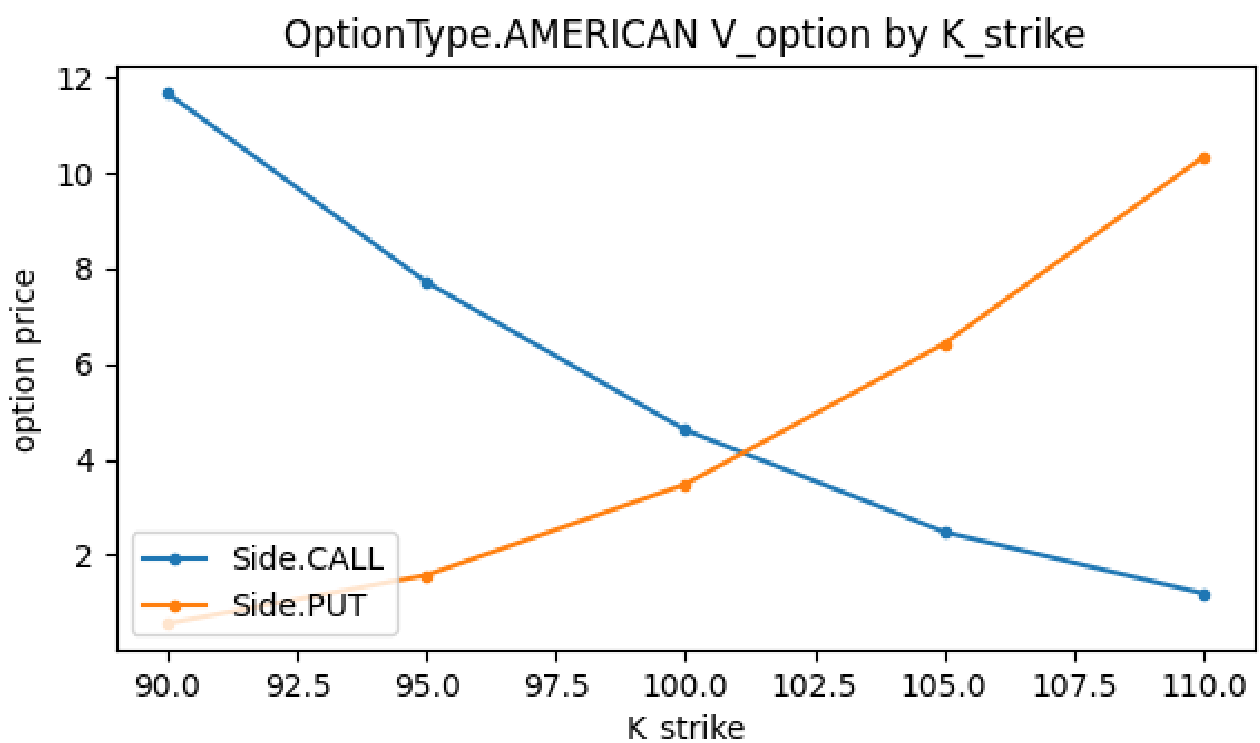
110.0

Q20.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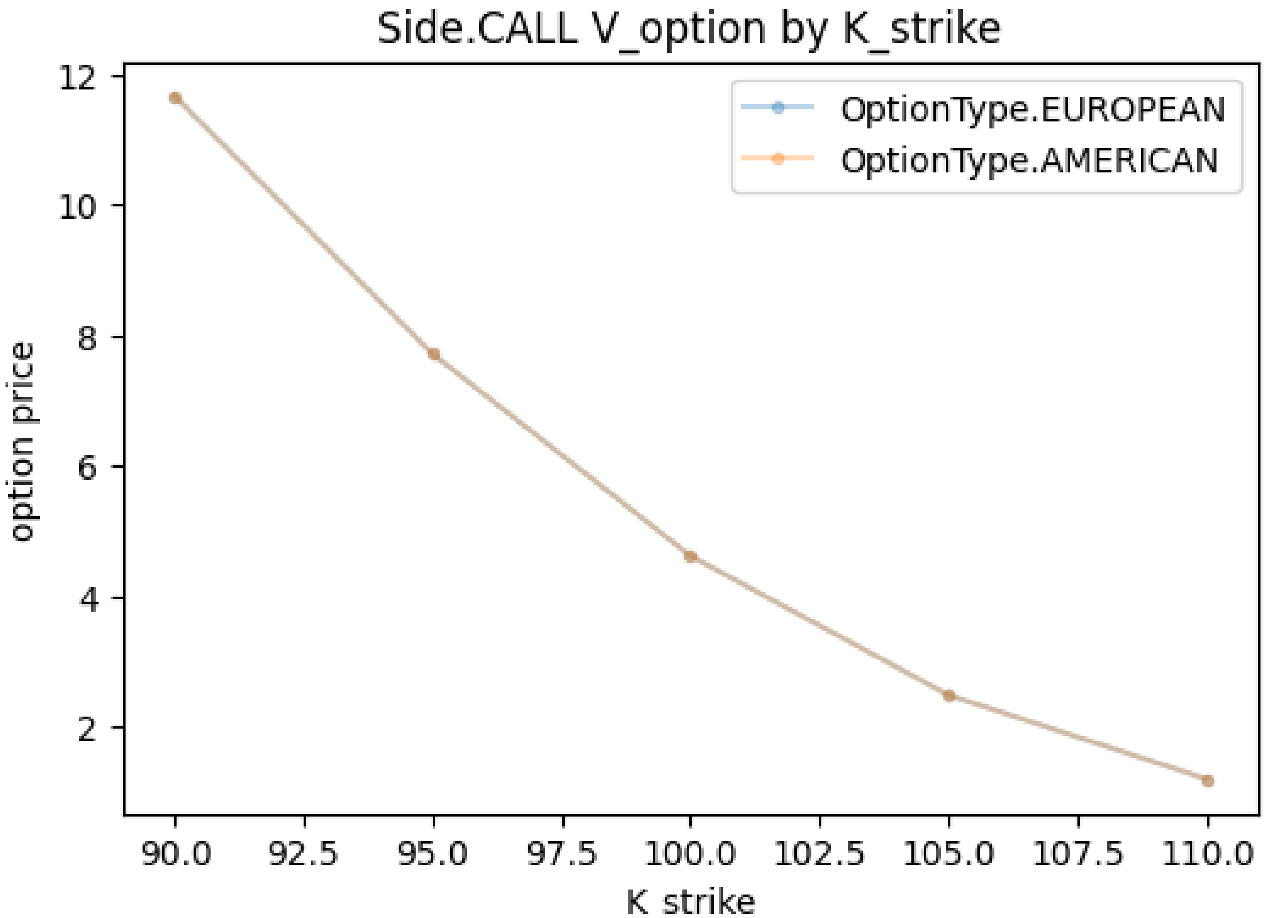
Q21.



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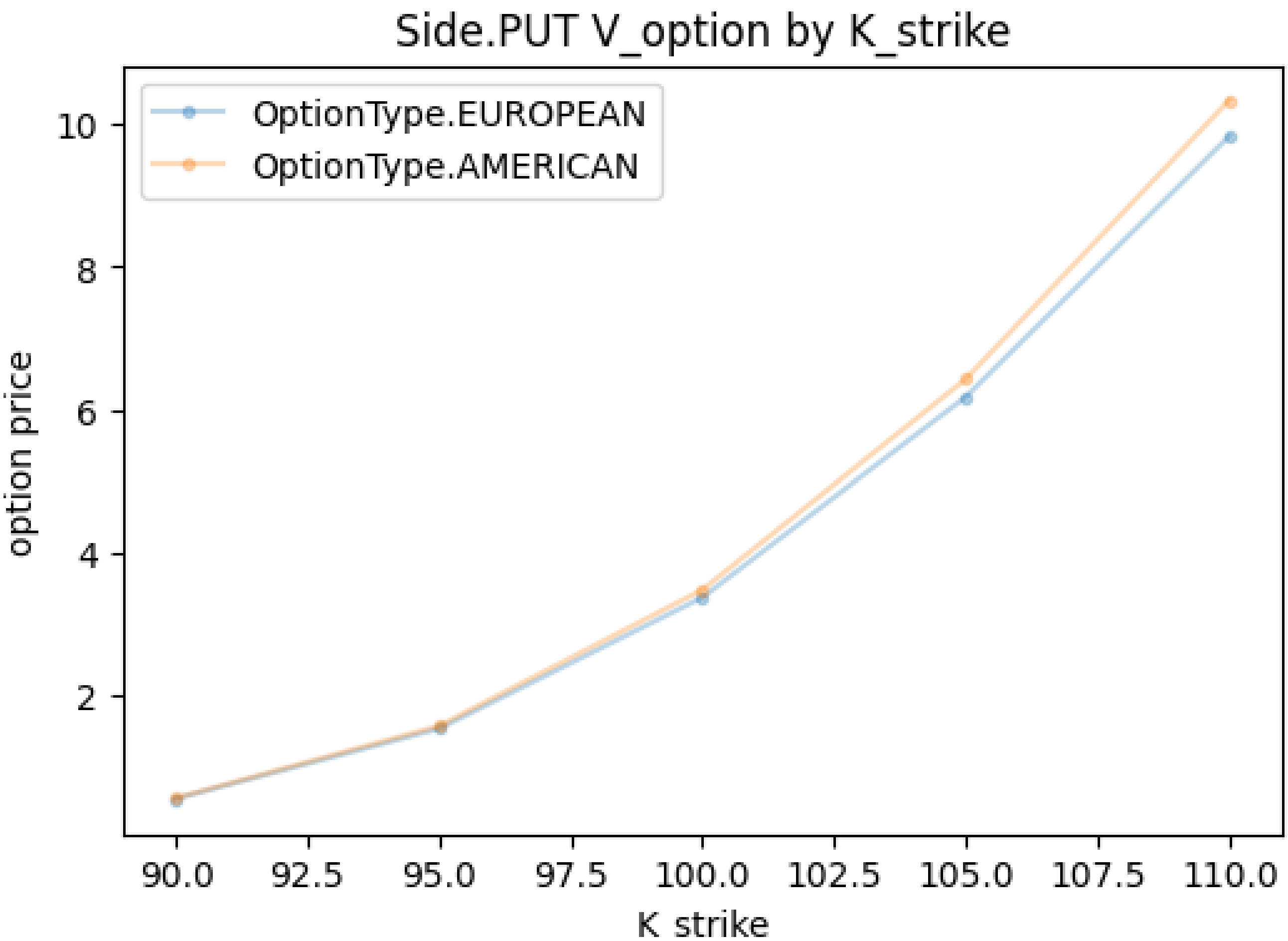
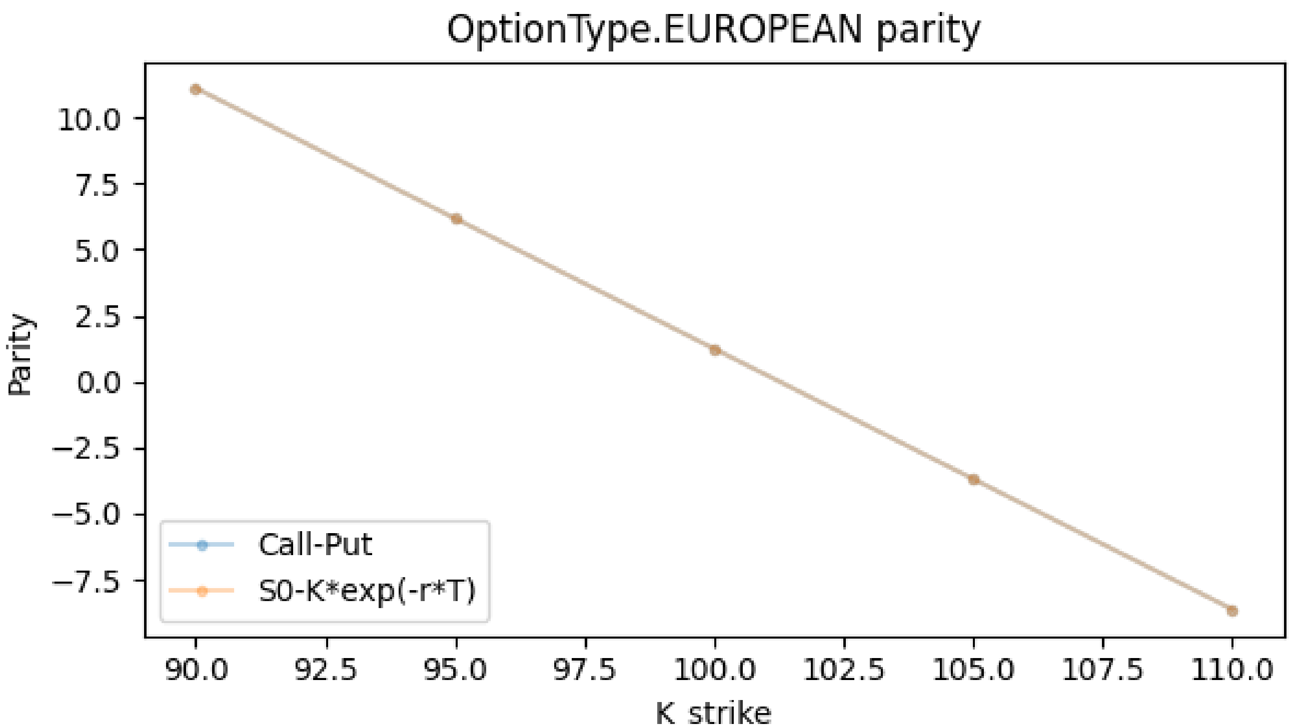
110.0

Q22.

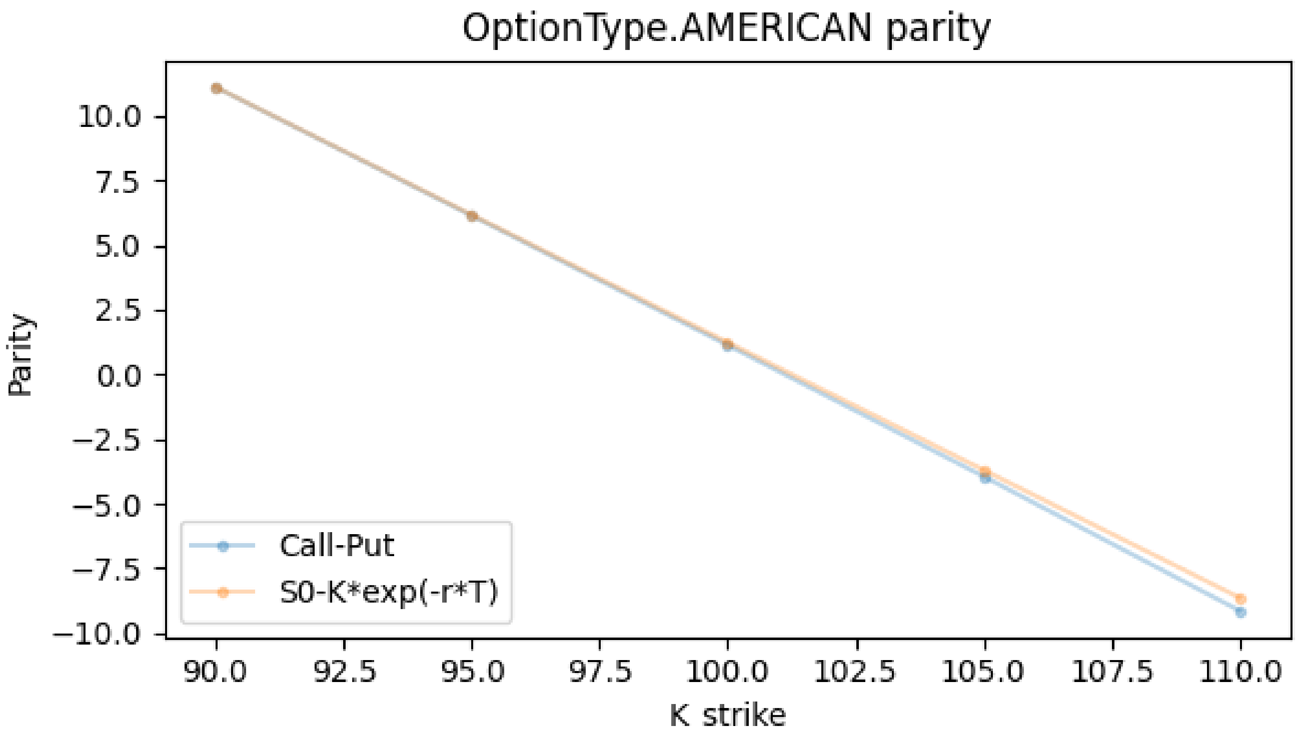


Q23. Recall the parity formula: $V^{CALL} - V^{PUT} = S_0 - Ke^{-rT}$

As shown in below figure representing the left hand side and the right hand side, the equation holds within the numerical precision error ($1e^{-9}$). This reaffirms our statement for Q1 where the parity equation holds for the European options.



Q24. As shown in below figure representing the left hand side and the right hand side, the two terms diverge significantly at the high strike price (K) - OTM for call, ITM for put - where the chance of early exercise of the American option prevails. Hence the parity equation does not hold but shows inequality as suggested in Q4.



Dynamic hedging process of options

Note, the notional in this demo means outflow when is positive, and inflow when is negative

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Q25-a. From the binomial tree, we got 13.82 with N=3

Q25-b. Delta is defined as the option price sensitivity to the underlying price $\Delta := \frac{\partial V}{\partial S}$, and is the price change of up/down state of the next option over that of underlying's in the binomial tree model. Since the Delta is the linear sensitivity, the Delta becomes the number of stocks we need to position (via long for positive or short for negative) the underlying per option to hedge against the price change of the option. Iterating this hedge positioning is termed as dynamic delta-hedging process.

Suppose the stock follows the path of $\{u, d, d\}$ in the binomial tree model and we are shorting the put, then we get the path of Stock, Option, Delta, Cash-flow (to hedge) paths as below:

Time	0	1	2	3
Stock	\$180	\$199.34	\$180	\$162.54
Option	- \$13.82	- \$5.01	- \$9.88	- \$19.46
Delta	0.47	0.24	0.53	N/A
Cash-flow (C)	- \$85.06	+ \$45.89	- \$51.57	+ \$105.42
Discount (DCF)	1.	0.997	0.993	0.990

The present value of cash-flows at the inception is $PV_{t=0}[\sum_t DCF_t \cdot C_t] = +\$13.82$, which matches with the option price in opposite sign. This shows the Arbitrage-free property of the delta-hedge process at the binomial modelled dynamics process - we recieve cash by selling the put but the cost of hedging by shorting the stock charged us the same amount at the risk-free rate funding condition.

At t=0, we short 0.47 lots of stocks hence receive $\$180 * 0.47$ amount of cash. At t=1, we decrease the short position to 0.24, hence had $\$199.34 * (0.24 - 0.47)$ amount of cash-flow (outflow) to cover the stock to clear shorts. At t=2, we increase the short position to 0.53, hence receiving $\$180 * (0.53 - 0.24)$ of cash. At t=3, the option is expired hence we clear all the position by paying $\$162.54 * 0.53$ cash to cover short-stock, as well as pay the option payoff by $\$(182 - 162.54)$.

Q-26. Suppose the stock follows the path of $\{u, d, d\}$ in the binomial tree model and we are shorting the put, then we get the path of Stock, Option, Delta, Cash-flow (to hedge) paths as below:

Time	0	1	2	3
Stock	\$180	\$199.34	\$180	\$162.54
Option	- \$13.98	- \$5.01	- \$9.88	- \$19.46
Delta	0.48	0.24	0.53	N/A
Cash-flow (C)	- \$86.56	+ \$47.56	- \$51.57	+ \$105.42
Discount (DCF)	1.	0.997	0.993	0.990

The present value of cash-flows at the inception is $PV_{t=0}[\sum_t DCF_t \cdot C_t] = +\$13.98$, which matches with the option price in opposite sign and the Arbitrage-free property of the binomial tree pricing model holds with the dynamic delta hedging process as like in the European put option's case.

The different at the delta (hence the cash-flow) paths compared to that of the European put's stems from the existence of optimal early-excercise. Higher odds of payout (as the put maker) making the price of up state of the option higher, causing higher delta for the American option. This amplified magnitude of Delta along the path incurs higher accumulated cost to maintain the dynamically risk-neutral hedge position, ended up paying more net cash-flow than the European put's.

Q-27. For the Asian option, the risk-neutral transition probability holds as $p(up) = \frac{e^{rt}-d}{u-d}$ as the dynamic delta heding is the Markovian process. The only difference with that of the European option for the hedging process is the terminal payoff which doesn't invalidate the Markovian property of the hedging process.

From our example dynamics setting, we have

- dt = 0.167
- u = 1.107
- d = 0.903
- dcf = 0.997
- $P_u = \frac{1/dcf-d}{u-d} = 0.491$
- $P_d = 1 - P_u = 0.509$

In put long holder's perspective, the valuation for each step backward from $\{u, d, d\}$ to $\{\}$ is calculated as follows:

- udd
  - $V_{udd} = K - Savg_{udd} = 182 - (180 + 199.34 + 180 + 162.54)/4 = 1.53$ as Asian put payoff
  - $D_{udd} = N/A$ as is at the final state
- ud
  - $V_{ud} = (V_{udd}P_d + V_{udu}P_u)dcf = (1.53 \cdot 0.51 + 0.0 \cdot 0.49) \cdot 0.997 = 0.78$
  - $D_{ud} = (V_{udu} - V_{udd})/(S_{udu} - S_{udd}) = (0.0 - 1.53)/(199.34 - 162.53523) = -0.04$
- u
  - $V_u = (V_{ud}P_d + V_{uu}P_u)dcf = (0.78 \cdot 0.51 + 0.0 \cdot 0.49) \cdot 0.997 = 0.39$
  - $D_u = (V_{uu} - V_{ud})/(S_{uu} - S_{ud}) = (0.0 - 0.78)/(220.76 - 180) = -0.02$
- .

- $V_{.} = (V_d P_d + V_u P_u)dcf = (14.59 \cdot 0.51 + 0.39 \cdot 0.49) = 7.60$
- $D_{.} = (V_u - V_d)/(S_u - S_d) = (0.39 - 14.59)/(199.34 - 162.54) = -0.39$

Hence in put short holder's perspective - value and sensitivity measured in opposite sign to that of the long holder's -, the cash-flow til expiry with the dynamic delta hedge process becomes:

Time	0	1	2	3
Transition	-	u	d	d
Stock	\$180	\$199.34	\$180	\$162.54
Option	-\$7.60	-\$0.39	-\$0.78	-\$1.53
Delta	0.39	0.02	0.04	N/A
Cash-flow (C)	- \$69.44	+ \$73.10	- \$4.06	+ \$8.29
Discount (DCF)	1.	0.997	0.993	0.990

, where the PV(t=0) of the cash-flow is +\$7.60 that matches with the option price at inception but in opposite sign. We paid for the delta hedge - the only risk modelled in the binomial tree model with static volatility (hence fixed up, down multiplier as well as transition probabilities) - as much as we earnt from the short, hence the arbitrage-free property holds for the price dynamics of the Asian option.